Section 9.4

Polar Coordinates:

A familiar way is to locate a point in the rectangular coordinate system denoted by \((x, y)\). Another coordinate system, called a polar coordinate system is introduced.

In the polar coordinate system, a point \(O\) called pole or origin is fixed and instead of a ray called polar axis is constructed through \(O\).

\( \theta \) - angle measured counterclockwise from initial position to terminal position.

\( r \) - distance from \(O\) to \(P\).

Each point \((x, y)\) in the rectangular coordinate system has a unique position. In the polar coordinate system, it is not true.

If \(r\) is negative, we locate the point on the ray from the pole in the opposite direction of the terminal side of the angle \(\theta\) at distance \(|r|\) from the pole.

Consider \(P\) with polar coordinates \((2, \pi/4)\). The same point \(P\) can be assigned the polar coordinates \((-2, 5\pi/4)\).
Coordinate Conversion

To convert from polar to rectangular coordinates and vice versa:

\[ x = r \cos \theta \quad y = r \sin \theta \]
\[ \tan \theta = \frac{y}{x} \quad x^2 + y^2 = r^2 \]

Example 1: Polar to rectangular,

(a) \((r, \theta) = (2, \pi)\)
\[ x = 2 \cos \pi \quad y = 2 \sin \pi \]
\[ = -2 \quad y = 0 \]
The rectangular coordinates are \((-2, 0)\)

(b) \((r, \theta) = \left(\sqrt{3}, \frac{\pi}{6}\right)\)
\[ x = \sqrt{3} \cos \frac{\pi}{6} \quad y = \sqrt{3} \sin \frac{\pi}{6} \]
\[ x = \frac{\sqrt{3}}{2} \cdot \sqrt{3} = \frac{3}{2} \quad y = \frac{\sqrt{3}}{2} \]
The rectangular coordinates are \(\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)\)

Example 2: Rectangular to Polar

\((x, y) = (-1, 1)\)
\[ r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}, \quad \tan \theta = -1 \quad \theta = 185^\circ = \frac{3\pi}{4} \]
Polar coordinates: \(\left(\sqrt{2}, \frac{3\pi}{4}\right)\)

(b) \((x, y) = (2, 2)\)
\[ r = 2 \quad \tan^{-1} \theta = 45^\circ \quad \theta = \frac{\pi}{4} \]
The polar coordinates, \((2, \frac{\pi}{4})\)

Polar Graphs — Convert from polar equation to rectangular equation and then plot.

(a) \(r = 2\)
\[ x^2 + y^2 = 2^2 \]
The graph is a circle with center \((0, 0)\)

(b) \(\theta = \frac{\pi}{2}\)
\[ \tan \frac{\pi}{2} = \frac{y}{x} = 0 \quad y = x \tan \frac{\pi}{2} = 0 \]
The graph is a vertical line.
Area Length

To determine the distance traveled by the particle,

\[ S = \int_{a}^{b} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2} \, dt \]

Finding Arc Length.

Example: \( x = 5 \cos t - \cos 5t \)
\[ y = 5 \sin t - \sin 5t \]

The curve has sharp points at \( x = 0 \) and \( t = \frac{\pi}{2} \). Between these points, the curve is smooth and \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) are not zero. To find the total distance traveled, find the arc length of one portion and multiply by 4 to get the arc length.

\[ S = 4 \int_{0}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

\[ = 4 \int_{0}^{\frac{\pi}{2}} \sqrt{-5 \sin t + 5 \sin 5t)^2 + (5 \cos t - 5 \cos 5t)^2} \, dt \]

\[ = 2 \int_{0}^{\frac{\pi}{2}} \sqrt{2 - 2 \cos 4t} \, dt = 2 \int_{0}^{\frac{\pi}{2}} \sqrt{2(1 - \cos 4t)} \, dt \]

\[ = 2 \int_{0}^{\frac{\pi}{2}} \sqrt{2 \sin^2 2t} \, dt = 2 \int_{0}^{\frac{\pi}{2}} 2 \sin 2t \, dt \]

\[ = -4 \cos 2t \bigg|_{0}^{\frac{\pi}{2}} = 40 \]
Area of a Surface of Revolution.

If a smooth curve given by \( x = f(t) \) and \( y = g(t) \) does not cross on an interval \( a \leq t \leq b \), the area \( S \) of the surface of revolution is given by:

1. \( S = 2\pi \int_{a}^{b} y(t) \sqrt{( \frac{dx}{dt} )^2 + ( \frac{dy}{dt} )^2} \, dt \)  
   Revolution about the \( x \)-axis.

2. \( S = 2\pi \int_{a}^{b} x(t) \sqrt{( \frac{dx}{dt} )^2 + ( \frac{dy}{dt} )^2} \, dt \)  
   Revolution about the \( y \)-axis.

These formulas are easy to remember.

\[
ds = \sqrt{( \frac{dx}{dt} )^2 + ( \frac{dy}{dt} )^2}
\]

1. \( S = 2\pi \int_{a}^{b} y \, ds \)  
   Revolution about the \( x \)-axis.

2. \( S = 2\pi \int_{a}^{b} x \, ds \)  
   Revolution about the \( y \)-axis.

Example:

\[x^2 + y^2 = 9\]

Find the area of the surface formed by revolving about the \( x \)-axis.

\( x = 3 \cos t \), \( y = 3 \sin t \).

\[
ds = \sqrt{ (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{( -3 \sin t )^2 + (3 \cos t)^2} = 3
\]

\[
S = 2\pi \int_{0}^{\pi/3} 3 \sin t \cdot 3 \, dt
\]

\[
= 18 \pi \left[ \sin t \right]_{0}^{\pi/3} = 18 \pi (1 - 0) = 18 \pi = 9\pi.
\]

Example: Find the area enclosed by the arc length of the

cylindrical arch of the parametric equation:

\[x = a (t - \sin t), \quad y = a (1 - \cos t), \quad 0 \leq t \leq \pi.
\]