Slope and Tangent Line:

The parametric curve \( x = f(t) \) and \( y = g(t) \) is called smooth if the derivatives \( f'(t) \) and \( g'(t) \) are continuous and never simultaneously zero. The slope \( \frac{dy}{dx} \) at \((x, y)\) is given by

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0
\]

\[
\frac{dy}{dx} = \frac{g'(t)}{f'(t)}, \quad f'(t) \neq 0
\]

The tangent line is vertical at any point where \( f'(t) = 0 \) but \( g'(t) \neq 0 \).

Example 1. Find \( \frac{dy}{dx} \) for the curve given by \( x = \sec t, \ y = \cos t \):

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{-\sin t}{\cos t} \right)
\]

\[
\frac{d^3y}{dx^3} = \frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) = \frac{d}{dx} \left( \frac{d}{dx} \left( \frac{-\sin t}{\cos t} \right) \right)
\]

Finding slope and concavity:

Example 2: \( x = \sqrt{t}, \ y = \frac{1}{4} \left( t^2 - 1 \right), \ t \geq 0 \)

Find the slope and the concavity:

\[
\frac{dx}{dt} = \frac{1}{2} t^{-1/2}, \quad \frac{dy}{dt} = \frac{1}{4} \cdot 2t = \frac{t}{2}
\]

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{t}{2}}{\frac{1}{2} t^{-1/2}} = t \cdot t^{1/2} = t^{3/2}
\]

To find second derivative:

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{t^{3/2}}{t^{-1/2}} \right) = \frac{3}{2} \cdot \frac{t^{1/2}}{t^{-1/2}} = 3 t
\]

at \((2, 3), \ 2 = t \Rightarrow t = 4 \)

\[
\frac{d^2y}{dx^2} \bigg|_{t=4} = 3 \cdot 4 = 12 \geq 0 \Rightarrow \text{graph is concave up}
\]
Example 3: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the cycloid

\[ x = a(t - \sin t) \quad \frac{dy}{dt} = a(1 - \cos t) \]
\[ \frac{dx}{dt} = a(1 - \cos t) \quad \frac{dy}{dx} = \frac{a \sin t}{1 - \cos t} \]

\[ \frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt} = \frac{\sin t}{1 - \cos t} \]

@ $t = \pi, 3\pi, \ldots$ (i.e., odd multiples of $\pi$)

\[ \frac{dy}{dx} = \frac{\sin \pi}{1 - \cos \pi} = 0 \quad \frac{1}{2} = 0 \]

The end points of the arches occur at even multiples of $\pi$, where both numerator and denominator are zero. These points are called cusps.

\[ \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy/\,dx}{\,dx/\,dt} \right) = \frac{d}{\,dx/\,dt} \left( \frac{\sin t}{1 - \cos t} \right) = \frac{(1 - \cos t) \cos t - \sin t (\sin t)}{(1 - \cos t)^2} \]

\[ = \frac{1}{a(1 - \cos t)^2} \]

Since the second derivative is negative for all negative multiples of $\pi$, $\Rightarrow \frac{d^2y}{dx^2} < 0$, the cycloid concaves downward.

Note: $\lim_{t \to 2n\pi} \frac{dy}{dx} = \lim_{t \to 2n\pi} \frac{\sin t}{1 - \cos t} = \lim_{t \to 2n\pi} \frac{\cos t}{\sin t} = \pm \infty$

$\Rightarrow$ tangent line is a vertical line at the cusp point.

Example 4: A curve with two tangent lines at a point.

\[ x = 2t - \pi \sin t \quad y = 2 - \pi \cos t \]

Choose at $(0, 2)$, find the equation of both tangent lines.

\[ t = \frac{\pi}{2} \]

\[ \frac{dy}{dx} = \frac{dy/\,dx}{\,dx/\,dt} = \frac{\pi \sin t}{2 - \pi \cos t} = \frac{\pi}{2 - \pi \cos t} \]

@ $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{\pi}{2}$

@ $t = -\frac{\pi}{2}$: $\frac{dy}{dx} = -\frac{\pi}{2}$

Eq. of line: $y - 2 = \frac{\pi}{2}x$
\[ x = 4y^2 - y \]

represent a parabola with a horizontal axis and vertex at \((-4, 3).\) The domain of the rectangular equation must be altered so that the graph matches the graph of the parametric equations.

**Adjusting Domain After Eliminating the Parameter**

\[
x = \frac{1}{\sqrt{t+1}} \quad y = \frac{t}{t+1} \quad t > -1
\]

\[
x^2 = \frac{1}{t+1}, \quad t = \frac{1-x^2}{x^2}
\]

\[
y = \frac{(1-x^2)/x^2}{x^2} = 1-x^2
\]

\[
y = 1-x^2
\]

The rectangular equation \(y = 1-x^2\) is defined for all values of \(x.\) But the parametric equation is defined when \(t > -1.\) The domain of \(x\) must be restricted to positive values.

**Using Trigonometric to Eliminate a Parameter**

\[
x = 2\sin \theta \quad y = 4\cos \theta \quad 0 \leq \theta \leq 2\pi
\]

\[
\frac{y}{2} = \cos \theta \quad \frac{y}{2} = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{x^2}{4}}
\]

\[
\frac{y^2}{16} = 1 - \frac{x^2}{4} \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{16} = 1
\]

The graph is an ellipse, centered at \((0, 0),\) with vertices \((0, 4)\) and \((0, -4)\) and minor axis of length 2. Orientation is counter clockwise.

**Note** (a) Parametric equations:

\[
x = 4 + a\cos \theta \quad y = 6\sin \theta \quad 0 \leq \theta \leq 2\pi
\]

Ellipse is traced counterclockwise, \(\frac{(x-4)^2}{a^2} + \frac{(y-6)^2}{b^2} = 1\)

(b) Parametric equations:

\[
x = 4 + 2\sin \theta \quad y = 6\cos \theta
\]

Ellipse is traced clockwise, \(\frac{(x-4)^2}{2^2} + \frac{(y-6)^2}{1^2} = 1\)

Parameteric equations tell the position, direction and speed at a given time.
Find Parametric Equations.

Determination of parametric equation for a graph is not unique. You may find more than one different parametric representation for a given graph.

**Example:** Finding Parametric Equations for a given graph.

\[ y = 1 - x^2 \]

Find the parametric equations representing the graph of \( y = 1 - x^2 \), using the following parameters.

(a) \( t = x \) and \( b) \ m = \frac{dy}{dx} = (x, y) \)

(a) \( x = t \) and \( y = 1 - t^2 \)

The parametric equations are \( x = t \) and \( y = 1 - t^2 \).

The orientation is from left to right.

(b) Express \( x \) and \( y \) in terms of parameter \( m \)

\[ m = \frac{dy}{dx} = -2x \]

\[ x = -\frac{m}{2} \]

\[ y = 1 - x^2 = 1 - \frac{m^2}{4} \]

**Note:** The curve is from right to left.

**Parametric Equation of a Cycloid**

The use of parametric equations \( x = x(t) \) and \( y = y(t) \) is most advantageous when elimination of parameter \( t \) is impossible or would lead to an equation \( y = f(x) \) that is more complicated than the original parametric equation.

The curve traced by a point \( P \) on the circumference of a circle of radius \( r \) rolling along a straight line is called a cycloid. Let \( \theta \) be the parameter to measure circular rotation.

Let \( P(x, y) \) begin at the origin. When \( \theta = 0 \), \( P \) is at the origin. When \( \theta = \pi \), \( P \) is at the maximum point \((\pi r, 2a)\).
When $\theta = 2\pi$, $P$ is back on the $x$-axis at $(2\pi a, 0)$. The distance the circle has rolled is $|OT|$ that is equal to the length of the circumference subtended by the angle TCP.

\[ \overrightarrow{CD} = a \cos \theta, \quad \overrightarrow{PQ} = a \sin \theta \]

\[ \overrightarrow{OT} = a \theta, \quad a \text{ is the radius of the circle.} \]

From the right angle triangle $PCA$:

\[ a \theta - x = a \sin \theta, \quad a - y = a \cos \theta \]

or

\[ x = a (\theta - \sin \theta) \]
\[ y = a (1 - \cos \theta) \]

The cycloid has sharp corners: @ $x = 2\pi ma$

\[ x' = a (1 - \cos \theta), \quad x'(2\pi m) = a - a \cos 2\pi m = 0 \]
\[ y' = a \sin \theta \]
\[ y'(2\pi m) = a \sin 2\pi m = 0 \]

Between these two points, the cycloid is smooth.