Section 9.5

Polar Area

To find the area bounded by polar graph, we use Riemann Sum. Instead of using rectangular areas, we sum the areas of circular sectors.

$A = \frac{1}{2} r^2 \theta$, where θ is the central angle of the sector measured in radians.

Area in Polar coordinates.

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

Example: Find the area of the polar region of one petal.

$r = 3\cos 3\theta \qquad -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

$$A = \frac{1}{2} \int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2 \, d\theta = \frac{9}{2} \int_{-\frac{\pi}{6}}^{\pi/6} \cos^2 3\theta \, d\theta$$

$$= \frac{9}{2} \int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2} \, d\theta = \frac{3\pi}{4}$$

Example: Find the area enclosed by four-leaved

$r = \cos 2\theta$

$$A = 8 \int_0^{\pi/4} \cos^2 2\theta \, d\theta = 8 \int_0^{\pi/4} \frac{1}{2} (1 + \cos 4\theta) \, d\theta$$

$$= 4 \left[\frac{\theta}{2} + \frac{\sin 4\theta}{4} \right]_0^{\pi/4} = \frac{\pi}{2}$$

Example: Find the area of the top half $0 \le \theta \le \pi$

$r = 1 + \cos\theta$

$$A = \frac{1}{2} \int_0^{\pi} (1+\cos\theta)^2 \, d\theta = \frac{1}{2} \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \frac{1}{2} \left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi} = \frac{3\pi}{4}$$

Ex Find the area of the region between the inner and outer loops of

$$r = 1 - 2\sin\theta$$

$$A_1 = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 4\sin\theta + 4\sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[\left(1 - 4\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right]\, d\theta$$

$$= \frac{1}{2}(2\pi - 3\sqrt{3}) = \pi - \frac{3\sqrt{3}}{2}$$

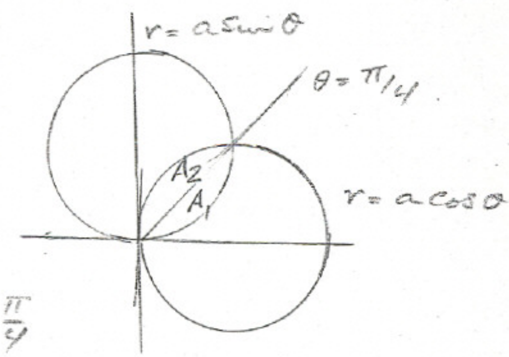
$$A_2 = \frac{1}{2}\int_{\frac{5\pi}{6}}^{13\pi/6} (1 - 2\sin\theta)^2\, d\theta \quad = 2\pi + \frac{3\sqrt{3}}{2}$$

$$A = A_2 - A_1 = 2\pi + \frac{3\sqrt{3}}{2} - \left(\pi - \frac{3\sqrt{3}}{2}\right)$$

$$= \pi + 3\sqrt{3} = 8.34$$

__Example__: Find the area of a region between two polar curves.

$$r = a\cos\theta \qquad r = a\sin\theta$$



Point of intersection:

$$a\cos\theta = a\sin\theta$$

$$\tan\theta = 1 \qquad \theta = \pi/4$$

R_1 is the region bounded by circle $r = a\sin\theta$ and the rays $\theta = 0$ and $\theta = \frac{\pi}{4}$

R_2 is the region bounded by circle $r = a\cos\theta$ and the rays $\frac{\pi}{4}$ and $\frac{\pi}{2}$.

A = Area of Region R_1 + Area of region R_2

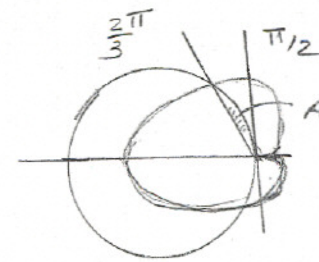
$$= \frac{1}{2}\int_0^{\pi/4} (a\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\pi/2} (a\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{8}(\pi - 2)$$

Example Find the area of the region bounded by

$r = -6\cos\theta$ and $r = 2 - 2\cos\theta$

Both the curves are symmetric with respect to the x-axis. We can work with the upper-half of the figure.



A_1 - region between the circle and the radial line $\theta = \frac{2\pi}{3}$.

Coordinates of the circle at the pole are $(0, \frac{\pi}{2})$. A_1, area can be obtained by integrating the circle from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$.

A_2 is the region between $\theta = \frac{2\pi}{3}$ and $\theta = \pi$ and $r = 2 - 2\cos\theta$

Integrate the second region, $r = 2 - 2\cos\theta$ between $\theta = \frac{2\pi}{3}$ to $\theta = \pi$.

$$\frac{A}{2} = \frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2 \, d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2 \, d\theta$$

$$= \frac{5\pi}{2}$$

$$A = 5\pi.$$

Example: Find the area between

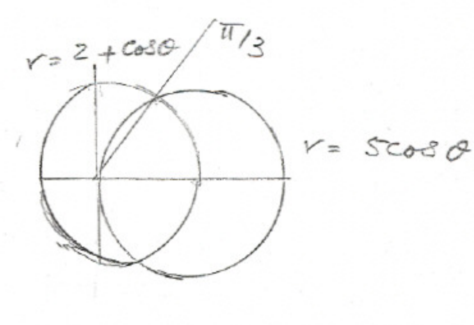
$r = 5\cos\theta$ and $r = 2 + \cos\theta$

Both the curves are symmetric with respect to the x-axis. We can find the area of the upper half and then multiply with 2 to find the total area.

$$5\cos\theta = 2 + \cos\theta$$

$4\cos\theta = 2$ and $\cos\theta = 1/2$

$$\theta = \frac{\pi}{3}$$



$$\frac{A}{2} = \frac{1}{2}\int_0^{\pi/3} (2 + \cos\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (5\cos\theta)^2 \, d\theta$$

$$A = \frac{43\pi}{12} - \sqrt{3} = 9.53.$$

Arc Length in Polar Form:

Let f be a function with continuous derivates on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$

$$S = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

or
$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example: Find the arc length from $\theta = 0$ to $\theta = 2\pi$ for

$$r = f(\theta) = 2 - 2\cos\theta$$

$$f'(\theta) = 2\sin\theta$$

$$S = \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta$$

$$= 2\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta = 2\sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2\theta/2}\, d\theta$$

$$= 4\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = 8\left[-\cos\frac{\theta}{2}\right|_0^{2\pi} = 16$$

Area of the Surface of Revolution:

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \quad \text{about polar axis}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \quad \text{about } \theta = \pi/2$$

Example

$r = f(\theta) = \cos\theta$

about the line $\theta = \frac{\pi}{2}$

$f'(\theta) = -\sin\theta$

$$S = 2\pi \int_0^{\pi} \cos\theta\,(\cos\theta) \sqrt{\cos^2\theta + (-\sin\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\pi} \cos^2\theta\, d\theta = \pi^2$$

Polar Equation of Conics:

The rectangular equations of ellipses, and hyperbolas are simplest when the origin lies at their center. In the conics, one of the foci is used as the origin for the coordinate system and the polar equations in conics are simplest when one of the foci is at the pole (origin).

Let D denote the directrix and F is a fixed point called focus. Let P be another point in the plane and e is eccentricity. The ratio of the distance $\overline{FP}$ and $\overline{PQ}$ equal e.

$$\frac{d(\overline{FP})}{d(\overline{PQ})} = e$$

If $e = 1$, the conics is a parabola

If $e < 1$, the conics is an ellipse

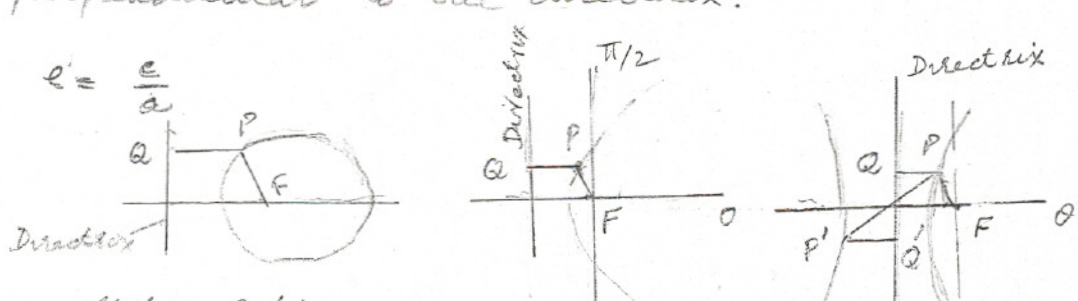
If $e > 1$, the conics is a hyperbola.

NOTE: 1. Equation of parabola is defined as the locus of all points P such that $d(\overline{PQ}) = d(\overline{FP})$. This means that $e = 1$

2. $e < 1$, It is an ellipse, the major axis is a line through F (focus) and is perpendicular to the directrix.

3. $e > 1$. The transverse axis is a line through focus F perpendicular to the directrix.

$e = \frac{c}{a}$

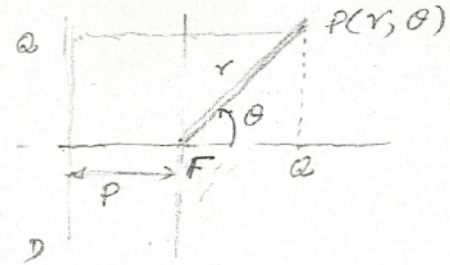


Ellipse $e < 1$
$\frac{PF}{PQ} < 1$

Parabola
$PF = PQ$

Hyperbola $\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

where c is the distance from the center to focus and a is the distance from the center to the vertex.



The focus is located at the pole and the directrix D is perpendicular to the polar axis.

Consider that the directrix D is pendicular to the polar axis located at a distance P from the left of the pole (or focus F).

$$\frac{(\overline{FP})}{(\overline{PQ})} = e$$

$$(\overline{FP}) = e\,\overline{PQ}$$

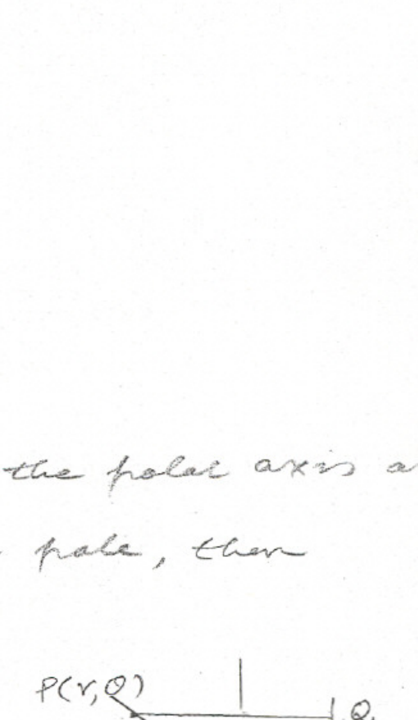
$$\overline{FP} = e(P + r\cos\theta)$$

$$r = e(P + r\cos\theta)$$

$$r = \frac{eP}{1 - e\cos\theta}$$

If the directrix is perpendicular to the polar axis at a distance P units to the right of the pala, then

$$r = \frac{eP}{1 + e\cos\theta}$$



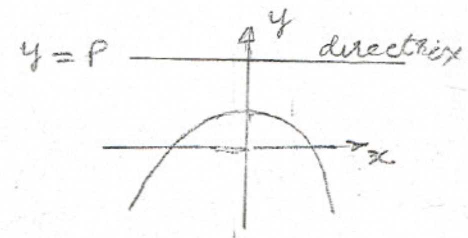
$$\overline{PQ} = P - r\cos\theta$$

$$\overline{FF} = r$$

$$r = e(P - r\cos\theta)$$

$$r = \frac{eP}{1 + e\cos\theta}$$

Four Types of Polar equations.

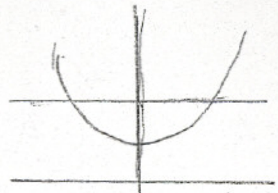


1. $r = \dfrac{eP}{1 + e\sin\theta}$

directrix is parallel to the polar axis at a distance P units above the pole.

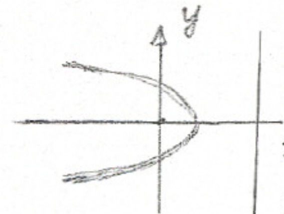
2. $r = \dfrac{eP}{1 - e\sin\theta}$

Directrix is parallel to the polar axis at a distance P units below the x-axis.



3. $r = \dfrac{eP}{1 + e\cos\theta}$

Directrix is perpendicular to the polar axis at a distance of P units to the right of the pole.



4. $r = \dfrac{eP}{1 - e\cos\theta}$

Directrix is perpendicular to the polar axis at a distance P units to the left of the pole.

Eccentricity : e

1. If $e = 1$, the conic is a parabola, the axis of symmetry is perpendicular to the directrix.

2. If $e < 1$, the conic is an ellipse. The major axis is perpendicular to the directrix.

3. If $e > 1$ the conic is a hyperbola, the transverse axis is perpendicular to the directrix.

Example. Sketch the graph:

$$r = \frac{4}{2 - \cos\theta}$$

Rewrite the equation in the form $r = \frac{eP}{1 - e\cos\theta}$

$$r = \frac{4/2}{1 - \frac{1}{2}\cos\theta} = \frac{2}{1 - \frac{1}{2}\cos\theta}$$

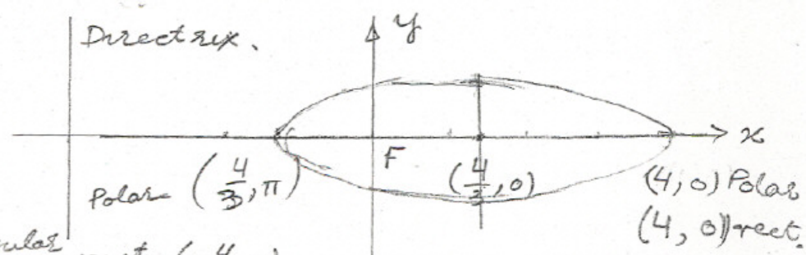
Since $e < \frac{1}{2}$, the conics is an ellipse.

$eP = 2$ or $\frac{1}{2}P = 2$; $P = 4$

Focus is at the pole and directrix is perpendicular to the polar axis 4 units to the left of the pole.

$\theta = 0$; vertex is $(4, 0)$.

$\theta = \pi$, vertex is $(\frac{4}{3}, \pi)$



midpoint is $\frac{4 - 4/3}{2} = \frac{4}{3}$

midpoint is $(\frac{4}{3}, 0)$ in rectangular coordinate system and $(\frac{4}{3}, 0)$ in polar coordinate.

$x = r\cos\theta$

$\frac{4}{3} = r\cos 0 \Rightarrow r = \frac{4}{3}$

$y = r\sin\theta$

$0 = r\sin 0 \Rightarrow r = 0$

$a = \frac{8}{3}$ distance from the center to the vertex

$b^2 = a^2 - c^2 = (\frac{8}{3})^2 - (\frac{4}{3})^2$

$b = \frac{4\sqrt{3}}{3}$

Convert polar equation to rectangular equation.

$(2 - \cos\theta)r = 4$

$2r = 4 + r\cos\theta$

$(2r)^2 = (4 + r\cos\theta)^2$

$4r^2 = (4 + x)^2$

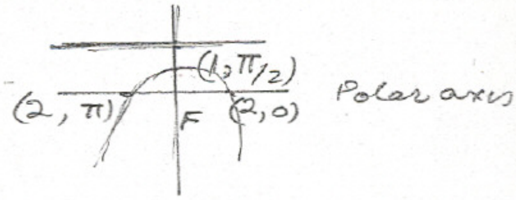
$4(x^2+y^2) = 16+8x+x^2$

$4x^2+4y^2-x^2-8x = 16$

This is the equation in the rectangular coordinate system.

Example $\quad r = \dfrac{6}{3+3\sin\theta}$

$r = \dfrac{6/3}{1+\sin\theta} = \dfrac{2}{1+\sin\theta}$.



$e = 1 \Rightarrow$ Parabola.

$eP = 2 \Rightarrow P = 2$

Directrix is parallel to the polar axis 2 units above the pole. Vertex is located at $(1, \frac{\pi}{2})$

Convert to rectangular coordinate.

$(3+3\sin\theta)\, r = 6$

$3r = 6 - 3r\sin\theta$

Square both sides.

$(3r)^2 = (6 - 3r\sin\theta)^2$

$9r^2 = (6-3y)^2$

$9(x^2+y^2) = (6-3y)^2$

$9x^2+9y^2 = 36 - 36y + 9y^2$

$9x^2 = 36 - 36y$

$y = +\frac{1}{4}x^2 + 1$

Example $\quad r = \dfrac{3}{1+3\cos\theta}$

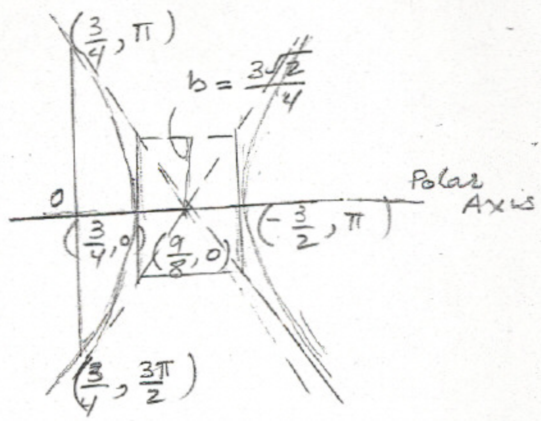
$e > 1$, hyperbola.

$eP = 3 \Rightarrow 3P = 3$ and $P = 1$

The directrix is perpendicular to the polar axis, 1 unit to the right of pole. The transverse axis is along the polar axis.

Vertices are located at $\theta = 0$ and $\theta = \pi$.

The vertices are $(\frac{3}{4}, 0)$, $(-\frac{3}{2}, \pi)$



$x = r\cos\theta$

$x = \frac{3}{4}\cos 0 = \frac{3}{4}$

$y = r\sin\theta$

$= \frac{3}{4}\sin 0 = 0$; $(\frac{3}{4}, 0)$

$x = r\cos\theta$

$x = -\frac{3}{2}\cos\pi = \frac{3}{2}$

$y = -\frac{3}{2}\sin\pi = 0$; $(\frac{-3}{2}, 0)$

midpoint $= \frac{1}{2}\left(\frac{3}{4} + \frac{3}{2}\right) = \frac{1}{2}\cdot\frac{9}{4} = \frac{9}{8}$, $\Rightarrow$

rectangular coordinates are $(\frac{9}{8}, 0)$

Polar coordinate: $\frac{9}{8} = r\cos 0 \Rightarrow r = \frac{9}{8}$

$\frac{9}{8} = r\sin 0 \Rightarrow \theta = 0$, $(\frac{9}{8}, 0)$ - Polar coordinates.

$c = \frac{9}{8}$ and $e = 3$

$\frac{c}{a} = 3$

$\frac{9/8}{a} = 3 \Rightarrow a = \frac{3}{8}$

$b^2 = c^2 - a^2 = \frac{81}{64} - \frac{9}{64} = \frac{72}{64} = \frac{9}{8}$ or $b^2 = c^2 - a^2 = (ea)^2 - a^2$

$= a^2(e^2 - 1)$

$b = \frac{3\sqrt{2}}{4}$

$b^2 = \left(\frac{3}{8}\right)^2(3^2 - 1) = \frac{9}{64}(8) = \frac{9}{8}$

<u>Example</u>: Graph $r = \dfrac{32}{3 + 5\sin\theta}$ Refer to Ex. 2 / P704

$r = \dfrac{32/3}{1 + \frac{5}{3}\sin\theta}$, $e = \frac{5}{3} > 1$ - hyperbola

$eP = 32/3$

$\frac{5}{3}P = 32/3$ or $P = 32/5$

Directrix is the line $y = 32/5$

The transverse axis of hyperbola lies on the line $\theta = \pi/2$

$(r, \theta) = (4, \pi/2)$, $(-16, 3\pi/2)$

$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right]$

$= 64$

$b = 8$

Convert to rectangular coordinate system.

$r(3+5\sin\theta) = 32$

$3r + 5\sin\theta \cdot r = 32$

$3r = 32 - 5r\sin\theta = 32 - 5y$

Square both sides.

$9r^2 = 1024 - 320y + 25y^2$

$9(x^2+y^2) - 25y^2 + 320y = 1024$

$9x^2 + 9y^2 - 25y^2 + 320y = 1024$

$9x^2 - 16y^2 + 320y^2 = 1024$