Section 9.4

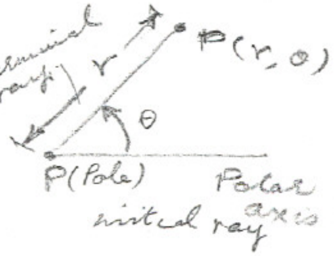
Polar Coordinates:

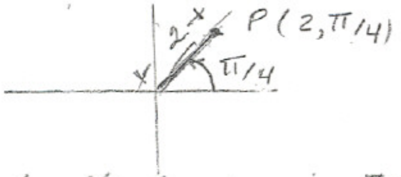
A familiar way is to locate a point in the rectangular coordinate system denoted by (x, y). Another coordinate system, called polar coordinate system is introduced.

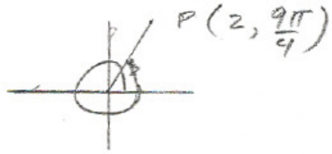
In the polar coordinate system, a point O called pole or origin is fixed and initial ray called polar axis is constructed through O.

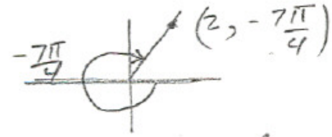
θ - angle measured counter clockwise from initial position to terminal position.

r - distance from O to P.



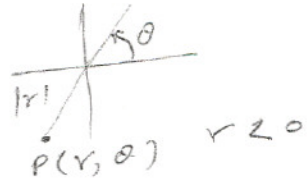




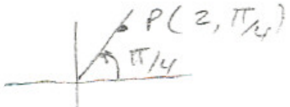


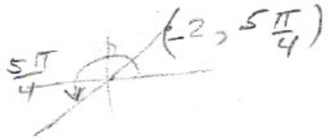
Each point (x, y) in the rectangular coordinate system has a unique position. In the polar coordinate system, it is not true.

If r is negative, we locate the point on the ray from the pole in the opposite direction of the terminal side of the angle θ at distance $|r|$ from the pole.



consider P with polar coordinates $(2, \frac{\pi}{4})$. The same point P can be assigned the polar coordinate $(-2, 5\frac{\pi}{4})$



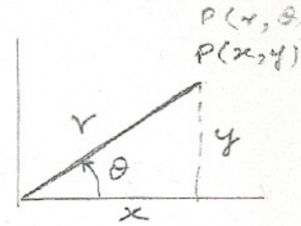


Coordinate Conversion

To convert from polar to rectangular coordinates and vice versa:

$x = r\cos\theta \qquad y = r\sin\theta$

$\tan\theta = \frac{y}{x}, \qquad x^2 + y^2 = r^2$



Example 1: Polar to rectangular.

(a) $(r, \theta) = (2, \pi)$

$x = 2\cos\pi, \quad y = 2\sin\pi$
$= -2, \quad y = 0$

The rectangular coordinates are $(-2, 0)$

(b) $(r, \theta) = (\sqrt{3}, \frac{\pi}{6})$

$x = \sqrt{3}\cos\frac{\pi}{6} \qquad y = \sqrt{3}\sin\frac{\pi}{6}$

$x = \frac{\sqrt{3}}{2}\sqrt{3} = \frac{3}{2} \qquad y = \frac{\sqrt{3}}{2}$

The rectangular coordinates are $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$

Example 2. Rectangular to Polar

$(x, y) = (-1, 1)$

$r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}, \qquad \tan\theta = -1 \qquad \theta = 135^\circ = \frac{3\pi}{4}$

Polar coordinates $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

(b) $(x, y) = (0, 2)$

$r = 2 \qquad \tan^{-1}\theta = \infty, \quad \theta = \frac{\pi}{2}$

The polar coordinates, $(2, \pi/2)$

Polar Graphs Convert from polar equation to rectangular equation and then plot.

(a) $r = 2 \qquad x^2 + y^2 = 2^2$

graph is a circle with center $(0, 0)$

(b) $\theta = \pi/3 \qquad \tan\frac{\pi}{3} = \frac{y}{x} \Rightarrow y = x\tan\frac{\pi}{3} = \sqrt{3}\,x$

(c) $r = \sec\theta \qquad x = r\cos\theta = \sec\theta \cdot \cos\theta = 1$

The graph is a vertical line.

Arc Length

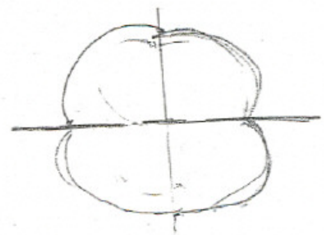
To determine the distance travelled by the particle.

$$S = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Finding Arc Length.

Example : $x = 5\cos t - \cos 5t$

$y = 5\sin t - \sin 5t$



The curve has sharp points at $x=0$ and $t = \frac{\pi}{2}$. Between these points the curve is smooth and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are not zero. To find the total distance travelled, find the arc length of one portion and multiply by 4 to get the arc length.

$$S = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= 4\int_0^{\pi/2} \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}\, dt.$$

$$= 20\int_0^{\pi/2} \sqrt{2 - \sin t \sin 5t - 2\cos t \cos 5t}\, dt$$

$$= 20\int_0^{\pi/2} \sqrt{2 - 2\cos 4t}\, dt = 20\int_0^{\pi/2} \sqrt{4\sin^2 2t}\, dt$$

$$= 20 \cdot 2\int_0^{\pi/2} \sin 2t\, dt = 40\int_0^{\pi/2} \sin 2t\, dt$$

$$= -20 \cos 2t \Big|_0^{\pi/2} = 40$$

Area of a Surface of Revolution.

If a smooth curve given by $x = f(t)$ and $y = g(t)$ does not cross on an interval $a \le t \le b$, the area S of the surface of Revolution is given by

1. $S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the x-axis

2. $S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the y-axis.

These formulas are easy to remember.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

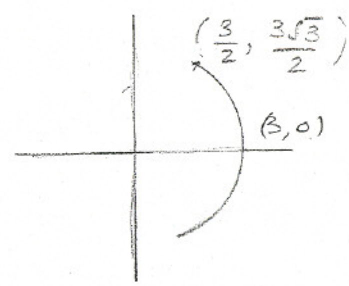
1. $S = 2\pi \int_a^b y\, ds$ – Revolution about the x-axis

2. $S = 2\pi \int_a^b x\, ds$ – Revolution about the y-axis

Example:

$x^2 + y^2 = 9$

Find the area of the surface formed by revolving about x-axis.



$x = 3\cos t \qquad y = 3\sin t.$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-3\sin\theta)^2 + (3\cos t)^2}$$

$$= 3$$

$$S = 2\pi \int_0^{\pi/3} 3\sin t \,.\, 3 \,.\, dt$$

$$= 18\pi \int_0^{\pi/3} \sin t\, dt = -18\pi \cos t \Big|_0^{\pi/3} = -18\pi\left(\tfrac{1}{2} - 1\right)$$

$$= 9\pi.$$

Example: Find the area under the arc length of the cycloidal arch of the parametric equation.

$x = a(t - \sin t), \qquad y = a(1 - \cos t) \qquad 0 \le t \le 2\pi.$