Section 9.2

Unil now, we have been representing graph by a single equation involving two variables. In this Section, three variables are used to represent a curve in the plane. Consider an object moving in the coordinate plane. The motion of the point can be described by $(x(t), y(t))$ at time t. The rectangular coordinate variables x and y are expressed as a function of a third variable, or parameter t. The parameter t is an independent variable.

Definition: Parametric Curve.

A parametric curve C in the plane is a pair of functions

$x = f(t)$ and $y = g(t)$ (1)

and x and y are continuous functions of the real number t. Each value of t determines a point $(f(t), g(t))$. The two equations in (1) are called parametric equations of the curve. and t is called the parameter.

Example Consider an object propelled into the air at an angle of $45°$ with an initial velocity of $48'/\text{sec}$. The parabolic path of the object travelled is

$y = -\dfrac{x^2}{72} + x$ rectangular equation.

By introducing a third variable, t, called a parameter, we can tell where the object was at a given point (x, y)

$x = 24\sqrt{2}\, t$
$y = -16t^2 + 24\sqrt{2}\, t$

are the parametric equations for x and y. @ $t = 0$, the object is at $(0,0)$ and @ $t = 1$, the object is at a point

$(24\sqrt{2},\ 24\sqrt{2} - 16)$

To plot a parametric equation, each point x and y is determined from a chosen value of parameter t. By plotting the resulting points as t increases in value, the curve

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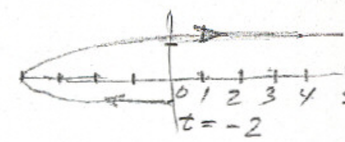
is traced in a specific direction. This is called orientation of the curve.

Example: Sketch a curve. $x = t^2 - 4$

$y = t/2 \qquad -2 \leq t \leq 3$

t	-2	-1	0	1	2	3
$x = t^2 - 4$	0	-3	-4	-3	0	5
$y = \frac{t}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$



The arrow on the curve indicates the orientation as t increases

Note: The parametric equation represent the graph that are more general than the graph of the function.

Consider: $x = 4t^2 - 4$ and $y = t \qquad -1 \leq t \leq \frac{3}{2}$

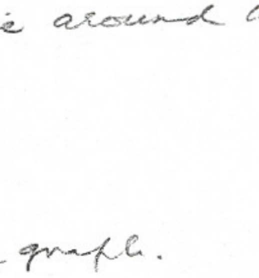
The graph is same as the previous graph. The second graph has different parametrization.

Different parametric representations can be used to represent different speed at which the object travels along a given path.

Example: Determine the graph of the curve

$x = \cos t\ ,\ y = \sin t \qquad 0 \leq t \leq 2\pi$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$



Every point of the graph lies on the circle with equation $x^2 + y^2 = 1$

As t travels from 0 to 2π, The point $(\cos t, \sin t)$ begins at $(1,0)$ and travels counter clockwise around the circle ending at $(1,0)$ when $t = 2\pi$.

$x = \cos t\ ,\ y = \sin t \qquad -\pi < t < \pi$

has different parametrization of the same graph.

Eliminating the parameter to determine the graph of the parametric curve.

$x = t^2 - 4 \quad —— (1)$

$y = t/2 \quad —— (2)$

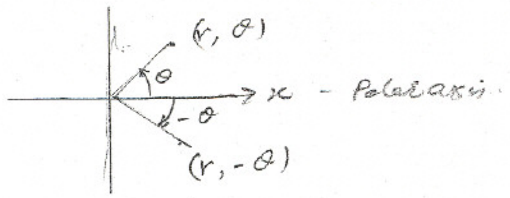
substitute $t = 2y$ from eq (2) in eq. (1). $\quad x = 4y^2 - 4$

The method of converting a polar equation to a rectangular eq. for graphing is not essential. We can set up a table that lists several points on the graph.

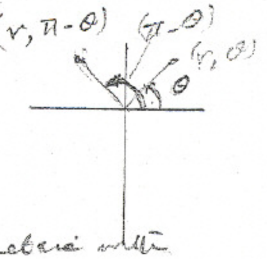
TESTS FOR SYMMETRY

Symmetry with respect to the polar axis (x-axis)

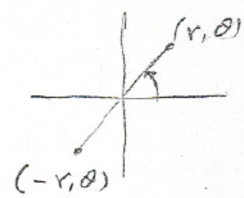
1. Replace θ with $-\theta$ in the polar equation. If an equivalent equation results, the graph is symmetric with respect to the polar axis.



Symmetric with respect to the polar axis



symmetric with respect to the line $\theta = \pi/2$



symmetric with respect to pole.

2. Symmetry with respect to the line $\theta = \frac{\pi}{2}$ (y-axis)

   Replace θ with $\pi - \theta$ in the polar equation. If the equivalent equation results, the graph is symmetric with respect to the line $\theta = \pi/2$.

3. Symmetry with respect to Pole - (origin)

   Replace r with $-r$. If the equivalent equation results, the graph is symmetric with respect to the pole.

Example: Graph $r = 1 + 2\cos\theta$

Check symmetry

1. Polar axis: Replace θ with $-\theta$

   $r = 1 + 2\cos(-\theta) = 1 + 2\cos\theta$ - Graph is symmetric with respect to polar axes.

2. The line $\theta = \frac{\pi}{2}$. Replace θ with $\pi - \theta$

   $r = 1 + 2\cos(\pi - \theta) = 1 - 2\cos\theta$ - Test fails. Graph is not symmetric with respect to $\theta = \frac{\pi}{2}$ (or y-axis)

3. The Pole: Replace r with $-r$.

   $-r = 1 + 2\cos\theta$ - test fails. not symmetric with respect to pole

Make a table for different values of θ and calculate the corresponding values of r. Due to symmetry with respect to polar axis, we need to determine the values of r for $\theta = 0$ to $\theta = \pi$.

θ	$r = 1 + 2\cos\theta$
0	3
$\frac{\pi}{6}$	2.73
$\frac{\pi}{3}$	2
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	0
$\frac{5\pi}{6}$	-0.73
π	-1

Example: Graph $r = 2\cos 2\theta$

Check Symmetry.

1. Polar axis: $r = 2\cos 2(-\theta) = 2\cos 2\theta$ – Symmetric about polar axis

2. about $\theta = \frac{\pi}{2}$ $r = 2\cos 2(\pi - \theta) = 2\cos(2\pi - 2\theta)$

$= 2\cos 2\theta$ – Symmetric about the line $\theta = \frac{\pi}{2}$

3. The Pole: Since the graph is symmetric about the polar axis (x-axis) and $\theta = \frac{\pi}{2}$ (y-axis), the graph is symmetric about the pole.

NOTE: This illustrates that the symmetry conditions are sufficient, but they are not the necessary conditions for the symmetry.

	$r = 2\cos 2\theta$
0	2
$\frac{\pi}{6}$	1
$\frac{\pi}{4}$	0
$\frac{\pi}{3}$	-1
$\frac{\pi}{2}$	-2

Example: Graph $r^2 = 4\sin 2\theta$.

Check Symmetry:

1. Polar axis: $r^2 = 4\sin 2(-\theta) = 4\sin(-2\theta) = -4\sin 2\theta$.
   not symmetric about polar axis.

2. $\theta = \frac{\pi}{2}$; $r^2 = 4\sin 2(\pi - \theta) = 4\sin(2\pi - 2\theta)$
   $= -4\sin 2\theta$.
   not symmetric about $\theta = \frac{\pi}{2}$

3. The Pole: $(-r)^2 = 4\sin 2\theta$
   $r^2 = 4\sin 2\theta$ - symmetric about the pole.

θ	$r^2 = 4\sin 2\theta$	r
0	0	0
$\frac{\pi}{6}$	$2\sqrt{3}$	± 1.9
$\frac{\pi}{4}$	4	2
$\frac{\pi}{3}$	$2\sqrt{3}$	± 1.9
$\frac{\pi}{2}$	0	0

Slope and Tangent Lines

To find the slope in the polar form, consider that the parametric equation $r = f(\theta)$ is differentiable.

$x = r\cos\theta = f(\theta)\cos\theta$ $\qquad$ $y = r\sin\theta = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

$\frac{dx}{d\theta} \neq 0$ at (r, θ)

Solutions.

1. If $\frac{dy}{d\theta} = 0 \Rightarrow$ horizontal tangents, provided $\frac{dx}{d\theta} \neq 0$
2. If $\frac{dx}{d\theta} = 0 \Rightarrow$ vertical tangents, provided $\frac{dy}{d\theta} \neq 0$
3. If $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0 \Rightarrow$ no conclusion.

<u>Example</u>: Find horizontal and vertical tangents.

$r = \sin\theta \qquad 0 < \theta < \pi$

$x = r\cos\theta \qquad y = r\sin\theta$

$x = \sin\theta\cos\theta \qquad y = \sin^2\theta$

$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta, \; \cos 2\theta = 0$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ ... vertical tangents @

$r = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

$\left(\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

$\frac{dy}{d\theta} = 2\sin\theta\cos\theta, \quad \sin 2\theta = 0$

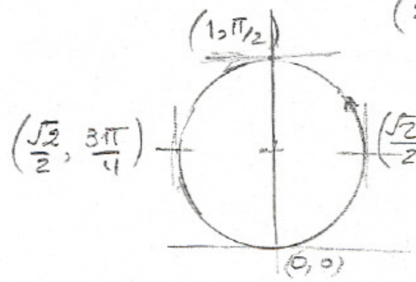
$\theta = 0, \frac{\pi}{2}$

$y = \sin^2 0 = 0, \quad (0, 0)$

$y = \sin^2\left(\frac{\pi}{2}\right) = \quad \left(1, \frac{\pi}{2}\right)$

horizontal tangents @

$(0,0), \quad \left(1, \frac{\pi}{2}\right)$



<u>Example</u> $\quad r = 2(1 - \cos\theta)$

Find horizontal and vertical tangents.

$x = r\cos\theta \qquad y = r\sin\theta$

$= 2(1-\cos\theta)\cos\theta \qquad y = 2(1 - \cos\theta)\sin\theta$

$x = 2\cos\theta - 2\cos^2\theta$

$\frac{dx}{d\theta} = -2\sin\theta + 4\cos\theta\sin\theta$

$= 2\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0, \quad \theta = 0,$

$\cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\frac{dy}{d\theta} = 2(1-\cos\theta)\cos\theta + 2\sin\theta\sin\theta$

$= 2[\cos\theta - \cos^2\theta + \sin^2\theta]$

$= -2(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -1/2, \quad \theta = \frac{2\pi}{3}, \frac{4\pi}{3},$

$\cos\theta = +1 \quad \theta = 0$

Vertical tangents occur @

$r = 2(1 - \cos 0), \; r = 0, \quad (0, 0)$

$r = 2\left(1 - \cos\frac{\pi}{3}\right), \; r = 1, \quad (1, \pi/3)$

$r = 2\left(1 - \cos\frac{5\pi}{3}\right), \; r = 1, \quad \left(1, \frac{5\pi}{3}\right)$

Horizontal tangents occur

$r = 2\left(1 - \cos\frac{2\pi}{3}\right) = 3, \; \left(3, \frac{2\pi}{3}\right)$

$r = 2\left(1 - \cos\frac{4\pi}{3}\right) = 3, \; \left(3, \frac{4\pi}{3}\right)$

Tangent Lines at the Pole

Suppose the graph $r = f(\theta)$ passes through the pole when $\theta = \alpha$

and $f'(\alpha) \neq 0$, $f(\alpha) = 0$

$$\frac{dy}{dx} = \frac{f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha}{f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha} = \frac{f'(\alpha)\sin\alpha}{f'(\alpha)\cos\alpha} = \tan\alpha$$

the line $\theta = \alpha$ is tangent to the graph $(0, \alpha)$

If $f(\alpha) = 0$, $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$

Example: $f(\theta) = 2\cos 3\theta$

For $\theta = \frac{\pi}{6}$, $\frac{\pi}{2}$ and $\frac{5\pi}{6}$, $f(\theta) = 0$

$f'(\theta) = -6\sin 3\theta$

$f'(\theta) \neq 0$ for $\theta = \frac{\pi}{6}, \frac{\pi}{2}$ and $\frac{5\pi}{6}$

Tangent lines at the pole occur at $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.