

Parabola with vertex at $(0,0)$

A parabola is defined as a set of points in a plane that are equidistant from a fixed line, called directrix, and a fixed point (focus) not on the directrix.

The line that passes through the focus and is perpendicular to the directrix is called axis of symmetry.

The midpoint of the line segment between the focus and the directrix is the vertex of the parabola.

Equation of a Parabola

Suppose the coordinates of the vertex of the parabola is $V(0,0)$ and the axis of symmetry is y -axis.

Focus lies on the axis of symmetry

distance $\overline{VP} = \text{distance } \overline{VD}$

Let $P(x,y)$ is any point on the parabola.

$$d(P,F) = d(P,D)$$

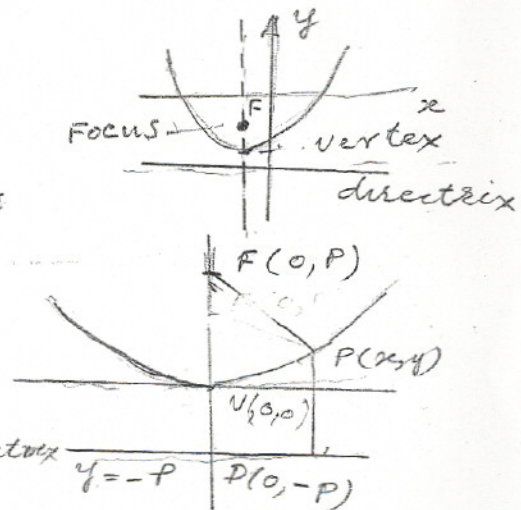
By the distance formula: $\sqrt{(x-0)^2 + (y-p)^2} = y+p$

Squaring both sides, $x^2 + y^2 - 2yp + p^2 = y^2 + 2yp + p^2$
 $x^2 = 4py$

Standard Form for the equation of Parabola with vertex at the origin.

Vertical Axis of Symmetry: $x^2 = 4py$, Focus is $(0,p)$ and directrix is $y = -p$

Horizontal Axis of Symmetry: $y^2 = 4px$, Focus is $(p,0)$ and directrix is $x = -p$.



NOTE: $x^2 = 4Py$

When $x^2 \geq 0 \Rightarrow 4Py > 0$

If $P > 0$, then $y \geq 0$ - Parabola opens up

If $P < 0$; then $y \leq 0$ - Parabola opens down.

(2) $y^2 = 4Px$

If $P > 0$, Parabola opens to the right

$P < 0$, Parabola opens to the left.

Example: Find the focus and the directrix.

$$y = -\frac{1}{2}x^2$$

$$x^2 = -2y \quad \text{compare } x^2 = 4Py$$

$$4P = -2 \text{ and } P = -\frac{1}{2}$$

Since P is negative, the parabola opens down.

Focus is located @ $(0, -\frac{1}{2})$ and directrix is $y = \frac{1}{2}$.

Example 2. Find the equation of the Parabola.

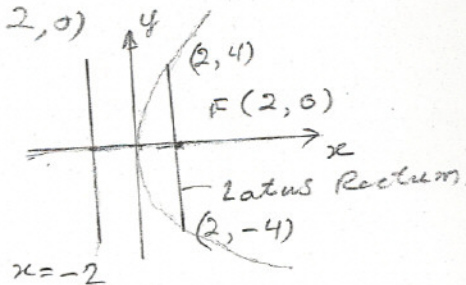
Vertex is at the origin and focus at $(+2, 0)$

Vertex is $(0, 0)$, $F(+2, 0)$, $P = -2$

$$y^2 = 4Px$$

$$y^2 = -8x$$

directrix: $x = -2$



Parabola with Vertex at (h, k)

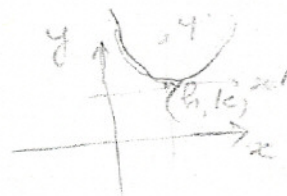
The equation of parabola with vertex at a point (h, k) can be found with the translation with the vertical and horizontal axes.

Consider a parabola with vertex at (h, k) .

$$x'^2 = 4Py'$$

$$(x-h)^2 = 4P(y-k)$$

Focus $F(h, k+P)$ and the equation of the directrix is $y = k - P$



the eq. of the parabola in the standard form is

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$$(y-2)^2 = 8(x-1)$$

Example

Find the focus:

$$y = -\frac{1}{2}x^2 - x + \frac{1}{2}$$

$$y - \frac{1}{2} = -\frac{1}{2}(x^2 + 2x)$$

$$= -\frac{1}{2}(x^2 + 2x + 1) + \frac{1}{2}$$

$$y - 1 = -\frac{1}{2}(x+1)^2$$

$$(x+1)^2 = -2(y-1)$$

Compare $(x-h)^2 = 4P(y-k)$

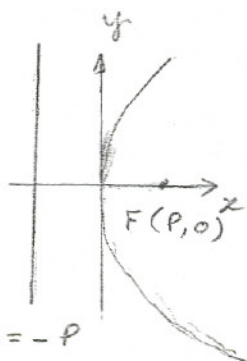
$h = -1, k = 1$, vertex $(-1, 1)$

$4P = -2$ and $P = -\frac{1}{2}$

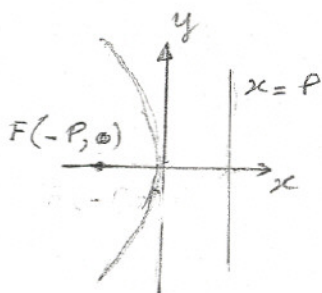
the eq. of the directrix is $y = k - P$

$$y = 1 + \frac{1}{2} = \frac{3}{2}$$

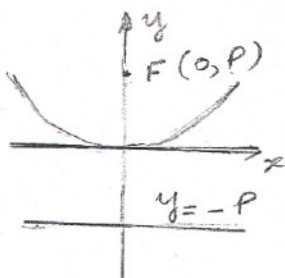
Focus, $(h, k+P) = (-1, 1 - \frac{1}{2}) = (-1, \frac{1}{2})$



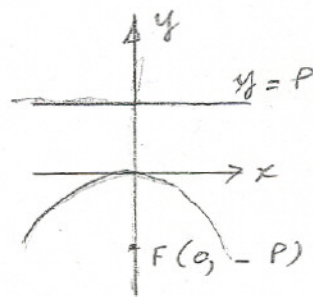
(a) $y^2 = 4Px$



(b) $y^2 = -4Px$



(c) $x^2 = 4Py$



(d) $x^2 = -4Py$

(a) and (b) Axis of symmetry is the x-axis.

(c) and (d). Axis of symmetry is the y-axis.

Example: Find the equation of parabola with vertex $(-2, 3)$ and focus at $(0, 3)$

The vertex $(-2, 3)$ and focus $(0, 3)$ both lie on the horizontal line $y = 3$ (axis of symmetry). The distance P from the vertex $(-2, 3)$ is $P = 2$.

$$(y-k)^2 = 4P(x-h)$$

$$(y-3)^2 = 4 \times 2(x+2)$$

$$(y-3)^2 = 8(x+2)$$

Horizontal Axis of symmetry.

$$y'^2 = 4P x'$$

$$(y-k)^2 = 4P(x-h)$$

The focus is $(h+P, k)$ and the equation of the directrix is

$$x = h - P.$$

Example: Find the eq. of the directrix, coordinates of the vertex

and the focus: $3x + 2y^2 + 8y - 4 = 0$

$$2y^2 + 8y = -3x + 4$$

$$2(y^2 + 4y + 4) - 8 = -3x + 4$$

$$2(y+2)^2 = -3x + 12$$

$$2(y+2)^2 = -3(x-4)$$

$$(y+2)^2 = -\frac{3}{2}(x-4)$$

Compare $(y-k)^2 = 4P(x-h)$

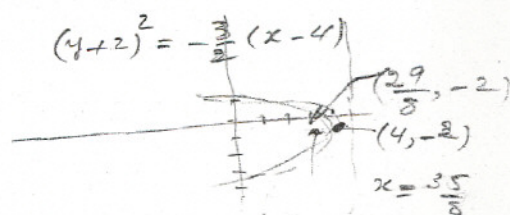
$$h = 4, k = -2, 4P = -\frac{3}{2} \text{ and } P = -\frac{3}{8}$$

Focus: $(4 - \frac{3}{8}, -2)$

$$(\frac{29}{8}, -2)$$

The equation of the directrix $x = h - P$

$$x = 4 + \frac{3}{8} = \frac{35}{8}$$



Example:

Equation of directrix $x = -1$ and $F(3, 2)$

Find the equation of parabola.

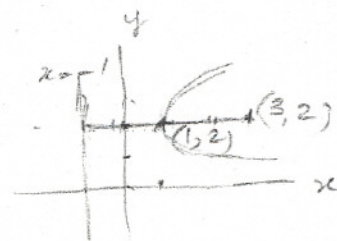
$$(y-k)^2 = 4P(x-h)$$

Vertex is located at the midpoint of $(3, 2)$ and $(-1, 2)$

$$h = \frac{3-1}{2} = 1, k = \frac{2+2}{2} = 2$$

$$(y-2)^2 = 4P(x-1)$$

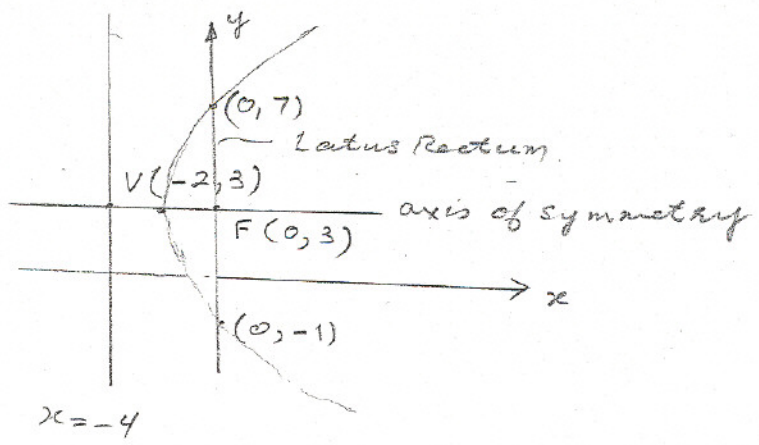
distance from vertex to focus = 2 $4P = 4(2) = 8$



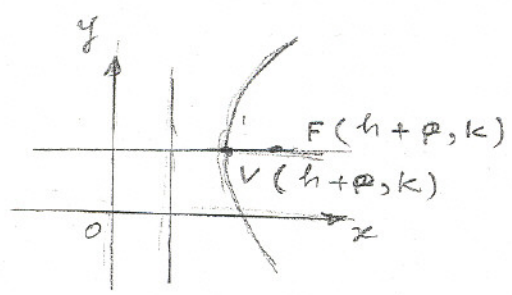
$\text{If } x=0, (y-3)^2 = 16$

$y-3 = \pm 4 \text{ or } y = -1 \text{ and } 7$

The points $(0, -1)$ and $(0, 7)$ define latus rectum; $x = -4$ is the directrix.

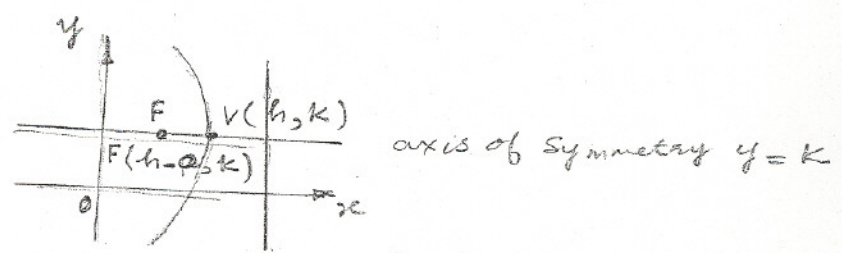


Parabolas with vertex at (h, k) ,



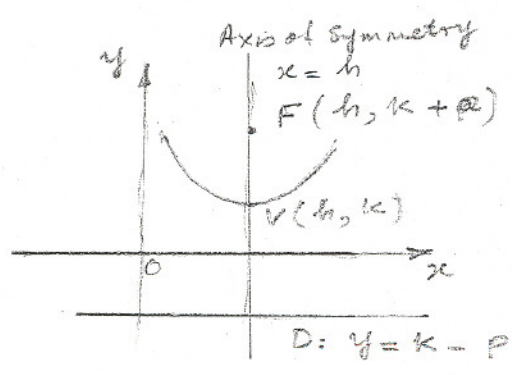
Directrix:
 $x = h - p$

(a) $(y - k)^2 = 4p(x - h)$

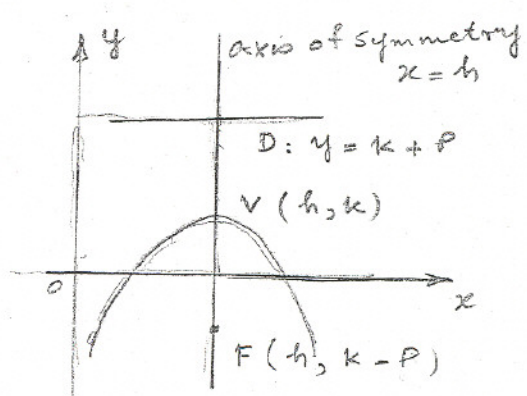


D: $x = (h + p)$

(b) $(y - k)^2 = -4p(x - h)$



(c) $(x - h)^2 = 4p(y - k)$



(d) $(x - h)^2 = -4p(y - k)$

Example: Find the length of the latus rectum of the parabola

$$x^2 = 4Py$$

The latus passes through the focus $(0, P)$ and is perpendicular to the y -axis. The coordinates of its end points, $(-x, P)$ and (x, P)

$$x^2 = 4Py, \quad y = P$$

$$x^2 = 4P^2, \quad \text{and } x = \pm 2P$$

$$S = \int_{-2P}^{2P} \sqrt{1 + y'^2} dx$$

$$= 2 \int_0^{2P} \sqrt{1 + \left(\frac{x}{2P}\right)^2} dx = \frac{1}{P} \int_0^{2P} \sqrt{4P^2 + x^2} dx$$

$$S = \frac{1}{2P} \left[x \sqrt{4P^2 + x^2} + 4P^2 \ln |x + \sqrt{4P^2 + x^2}| \right]_0^{2P}$$

$$= \frac{1}{2P} \left[2P \sqrt{4P^2 + 4P^2} + 4P^2 \ln |2P + \sqrt{4P^2 + 4P^2}| - 4P^2 \ln 2P \right]$$

$$= \frac{1}{2P} \left[2P \sqrt{8P^2} + 4P^2 \ln (2P + \sqrt{8P^2}) - 4P^2 \ln (2P) \right]$$

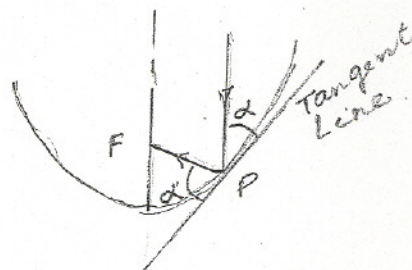
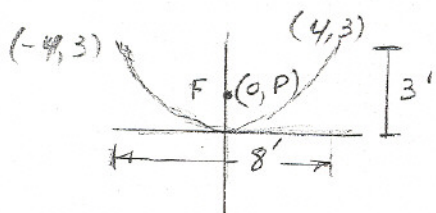
$$S = 2P \left[\sqrt{2} + \ln (1 + \sqrt{2}) \right] = 4.59P$$

Reflective Property

The line tangent to a parabola at P makes equal angles with the line through P and parallel to the axis of symmetry and the line PF , i.e., the line from P to the focus of the parabola.

Example

Find the location of receiver if the satellite dish is shaped like a paraboloid.



$$x^2 = 4Py$$

$$4^2 = 4P(3), \quad P = \frac{4}{3}$$

The receiver should be located at $P = \frac{4}{3}$ from the base of the dish and along the line of symmetry.

The Ellipse

An ellipse is a locus of all points in the plane where the sum of distances from two fixed points, called the foci is constant.

The foci are labeled F_1 and F_2 . The line passing through the foci is called major axis. The midpoint of the line segment joining the foci is called the center of the ellipse. The line perpendicular to the major axis and passing through the center is called minor axis.

Equation of an Ellipse, center $(0,0)$

Foci $(\pm c, 0)$. Major axis along x -axis.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b$$

$$\text{and } b^2 = a^2 - c^2$$

The ellipse is symmetric with respect to x -axis, y -axis and origin.

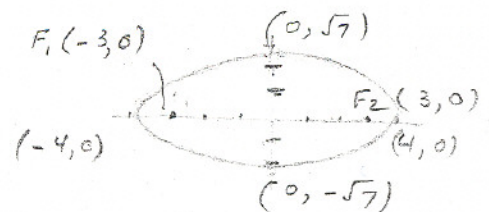
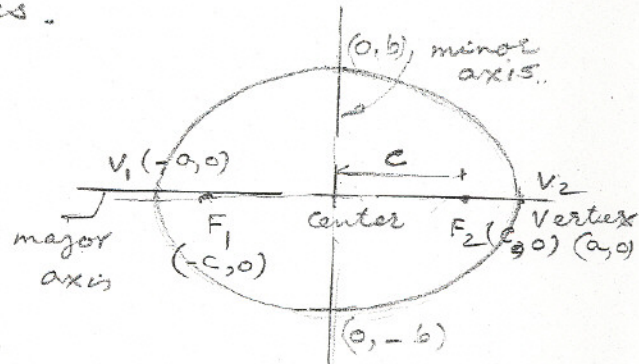
Example. Focus $(3,0)$ and vertex $(-4,0)$. Find the equation of an ellipse.

$$b^2 = a^2 - c^2 = 16 - 9 = 7$$

$$\frac{x^2}{16} + \frac{y^2}{7} = 1$$

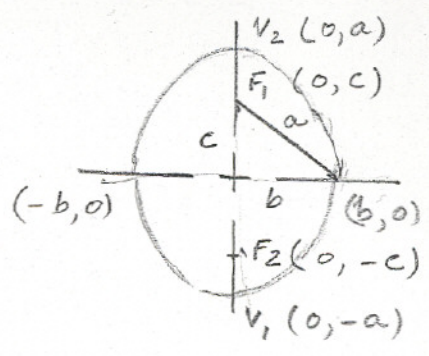
$$c^2 = 16 - 7 = 9, \quad c = \pm 3$$

The foci are $(-3,0)$ and $(3,0)$



Equation of an Ellipse: center at (0,0), Foci at (0, ±c). Major axis along y-axis.

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \text{where } b^2 = a^2 - c^2$$



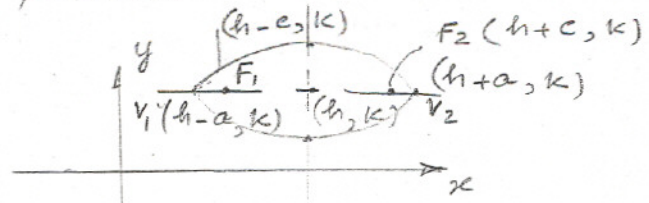
Example: One focus at (0, 2) and vertices @ (0, -3), (0, 3) Find the equation of an ellipse.

$$c = 2, a = 3, b = \pm\sqrt{9-4} = \pm\sqrt{5}$$

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Center at (h, k); Equation of an Ellipse

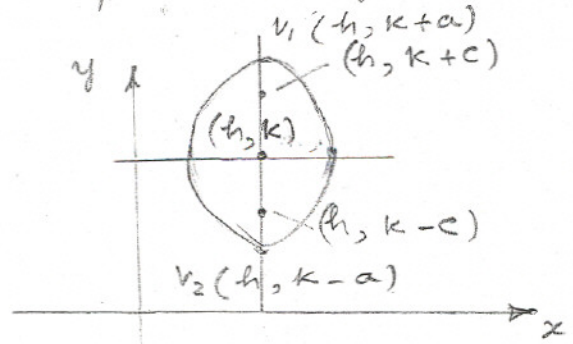
1. Major axis parallel to x-axis.



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$a > b, b^2 = a^2 - c^2$$

2. Major axis parallel to y-axis.



$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$a > b, b^2 = a^2 - c^2$$

Example:

$$4x^2 + y^2 - 8x + 4y - 8 = 0$$

Find centre, foci and vertices.

$$4x^2 - 8x + y^2 + 4y - 8 = 0$$

$$4(x^2 - 2x) + y^2 + 4y = 8$$

$$4(x^2 - 2x + 1 - 1) + y^2 + 4y + 4 - 4 = 8$$

$$4(x^2 - 2x + 1) - 4 + (y^2 + 4y + 4) - 4 = 8$$

$$4(x-1)^2 + (y+2)^2 = 16$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

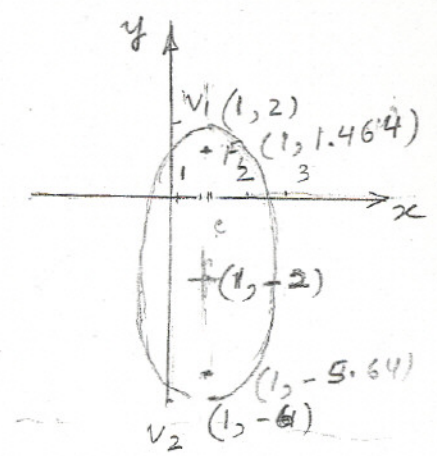
$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{4^2} = 1$$

$h = 1, k = -2$. Center $(1, -2)$

$$b = 2, a = 4, c^2 = a^2 - b^2$$
$$c^2 = 16 - 4 = 12$$

$$c = \pm 2\sqrt{3}$$

Foci; $F_1(1, -2 - 2\sqrt{3}), F_2(1, -2 + 2\sqrt{3})$

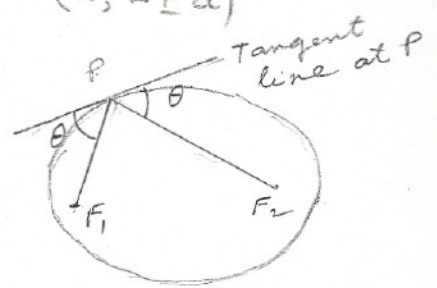


Reflective Property of an Ellipse.

Let P be a point on an ellipse. The tangent line to the ellipse at point P

makes an equal angles with the lines through P and the foci.

The reflective property of an ellipse states that the tangent line at P of an ellipse makes equal angle with two lines PF₁ and PF₂ from P to the foci, F₁ and F₂ of an ellipse.



Definition of Eccentricity of an Ellipse.

The eccentricity of an ellipse is given by the relationship

$$e = \frac{c}{a}$$

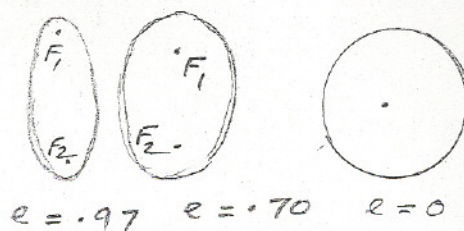
The concept of eccentricity is used to determine the ovalness of an ellipse.

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The larger the eccentricity $e < 1$, the more elongated the ellipse.

If $e = 0$, the equation is a circle.

The foci are located between the vertices and the center of an ellipse along the major axis.



$$0 < c < a$$

If the foci are close to the center and the ratio $\frac{c}{a}$ is small, then the ellipse is nearly circular.

For ellipse, $0 < e < 1$. Larger the ratio $e = \frac{c}{a}$, more elongated is an ellipse.

Example orbit of the sun is an ellipse with the sun as focus. The maximum, the maximum distance from the center of the sun is 94.56 million miles, and its minimum distance is 91.44 million miles. Find the major and minor axes of the earth's orbit and eccentricity.

$$a + c = 94.56$$

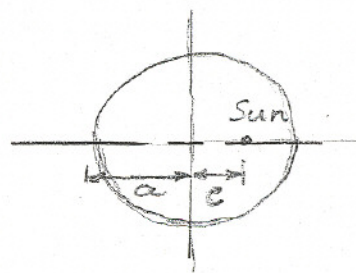
$$a - c = 91.44$$

$$c = 1.56, \quad a = 93.00$$

$$b = \sqrt{a^2 - c^2} = \sqrt{93^2 - 1.56^2} \approx 92.99$$

$$e = \frac{c}{a} = \frac{1.56}{93} = .017$$

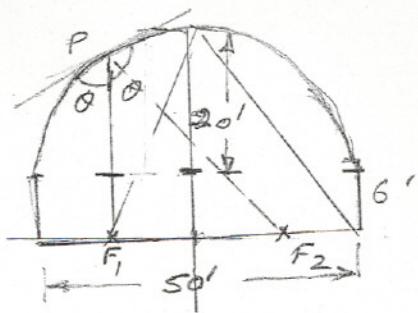
e is very close to zero, this means that earth is close to a circle.



Whispering Gallery:

The hall of a building has an elliptical ceiling. In a whispering gallery, two persons whispering standing at a focus of the ellipse can be heard by another person

Standing at the other focus, because the tangent line at P makes an equal angle with the lines PF_1 and PF_2 . i.e., The angle of incidence and angle of reflection are equal.

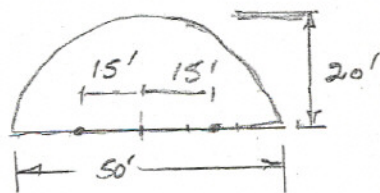


$$\frac{x^2}{25^2} + \frac{y^2}{20^2} = 1$$

$$a = 25, \quad b = 20; \quad c^2 = 25^2 - 20^2$$

$$c^2 = 225$$

$$c = 15$$



Circumference and Area of an Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$A = 4 \int_0^b \frac{b}{a} \sqrt{a^2 - x^2} dx$$

$$= \frac{4b}{a} \int_0^{\pi/2} a^2 \cos^2 \theta d\theta \quad \text{where } x = a \sin \theta$$

$$= 4ab \int_0^{\pi/2} \cos^2 \theta d\theta = 4ab \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) \Big|_0^{\pi/2}$$

$$A = \pi ab$$

It is not simple to find the circumference of an ellipse. Eccentricity is used to set up elliptic integrals to determine the circumference of an ellipse.

$$y = \frac{b}{a} \sqrt{a^2 - x^2}$$

$$\text{Arc length} = \int \sqrt{1 + y'^2} dx$$

Since the ellipse is symmetric about x - and y -axes, the circumference is equal to $4 \times$ arc length.

$$C = 4 \int_0^a \sqrt{1 + y'^2} dx$$

$$C = 4 \int_0^a \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}} dx$$

$$\text{Let } x = a \sin \theta$$

$$C = 4 \int_0^{\pi/2} \sqrt{1 + \frac{b^2 x^2}{a^2 \cos^2 \theta}} \cdot a \cos \theta d\theta$$

$$C = 4 \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$$

$$= 4 \int_0^{\pi/2} \sqrt{a^2(1 - \sin^2 \theta) + b^2 \sin^2 \theta} d\theta$$

$$= 4 \int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2) \sin^2 \theta} d\theta$$

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2}$$

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta$$

Such integrals do not have simple antiderivatives. Approximate methods are used to find antiderivatives.

Example: $\frac{x^2}{25} + \frac{y^2}{16} = 1$

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = \frac{25 - 16}{25} = \frac{9}{25}$$

$$C = 4 \times 5 \int_0^{\pi/2} \sqrt{1 - \frac{9 \sin^2 \theta}{25}} d\theta$$

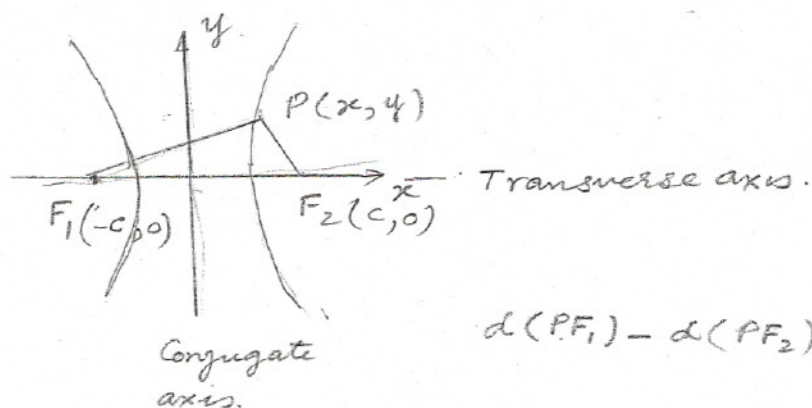
Applying Simpson's Rule, $n=4$

$$C = 20 \times \frac{\pi}{6} \times \frac{1}{4} [1 + 4(.9733) + 2(.9055) + 4(.8323) + .8]$$

$$\approx 28.36$$

The Hyperbola

A hyperbola is the set of all points (x, y) in the plane for which the difference from two fixed points called foci is constant.

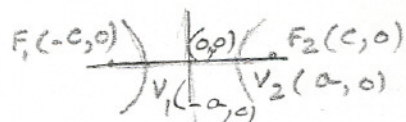


$$d(P, F_1) - d(P, F_2) = \text{constant}$$

The line through two foci intersects the hyperbola at two points called vertices. The line passing through the foci is called Transverse axis and the line through the midpoint of the vertices and perpendicular to the transverse axis is called conjugate axis.

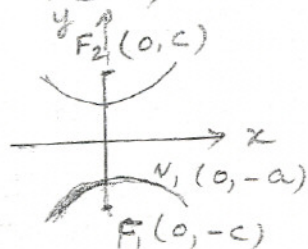
Standard Equation of Hyperbola: Center at $(0, 0)$, Foci at $(\pm c, 0)$ vertices at $(\pm a, 0)$. Transverse axis along x -axis.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{where } c^2 = a^2 + b^2$$



Center at $(0, 0)$, Foci at $(0, \pm c)$, vertices at $(0, \pm a)$. Transverse axis along the y -axis

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \quad \text{where } c^2 = a^2 + b^2$$



Asymptotes of a Hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y^2 = b^2 \left(\frac{x^2}{a^2} - 1 \right)$$

$$y^2 = \frac{b^2 x^2}{a^2} \left(1 - \frac{a^2}{x^2}\right)$$

$$y = \pm \frac{bx}{a} \sqrt{1 - \frac{a^2}{x^2}}$$

As $x \rightarrow \pm \infty$, $\frac{a}{x} \rightarrow 0$;

$$y = \pm \frac{bx}{a}$$

These lines are oblique asymptotes of hyperbola

Asymptotes of Hyperbola: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x$$

Example:

$$9x^2 - 4y^2 = 36$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

$$a=2, \quad b=3, \quad c^2 = a^2 + b^2 = 4 + 9 = 13$$

$$c = \pm \sqrt{13}$$

$$\text{asymptotes: } y = \pm \frac{b}{a}x = \pm \frac{3}{2}x$$

center (h, k)

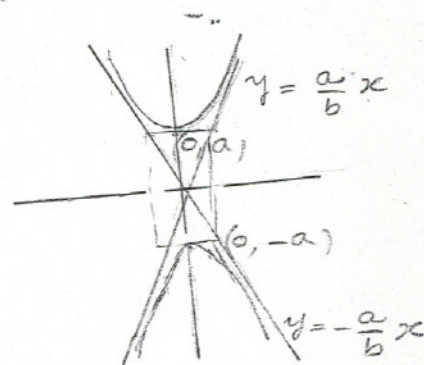
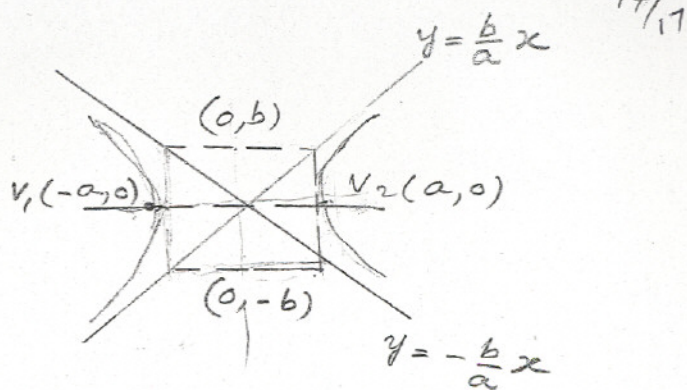
1. Equation of Ellipse, center (h, k) ; Transverse axis parallel to x -axis,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad c^2 = a^2 + b^2$$

Foci: $(h \pm c, k)$

vertices: $(h \pm a, k)$

asymptotes: $y - k = \pm \frac{b}{a}(x - h)$



2. Equation of Ellipse, center (h, k) , Transverse axis parallel to y -axis.

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Foci: $(h, k \pm c)$

Vertices: $(h, k \pm a)$

Asymptotes: $y - k = \pm \frac{a}{b}(x - h)$

Example: Find the vertices, foci and asymptote of hyperbola

$$4x^2 - 9y^2 - 16x + 54y - 29 = 0$$

$$4x^2 - 16x - 9y^2 + 54y = 29$$

$$4(x^2 - 4x) - 9(y^2 - 6y) = 29$$

$$4(x^2 - 4x + 2^2 - 2^2) - 9(y^2 - 6y + 3^2 - 3^2) = 29$$

$$4(x-2)^2 - 4 \cdot 2^2 - 9(y-3)^2 + 9 \cdot 3^2 = 29$$

$$4(x-2)^2 - 9(y-3)^2 = 29 + 16 - 81$$

$$4(x-2)^2 - 9(y-3)^2 = -36$$

$$9(y-3)^2 - 4(x-2)^2 = 36$$

$$\frac{(y-3)^2}{2^2} - \frac{(x-2)^2}{3^2} = 1$$

The coordinates of the center $(2, 3)$. The transverse axis is parallel to y -axis.

$$c^2 = a^2 + b^2 = 4 + 9 = 13$$

$$c = \pm \sqrt{13}$$

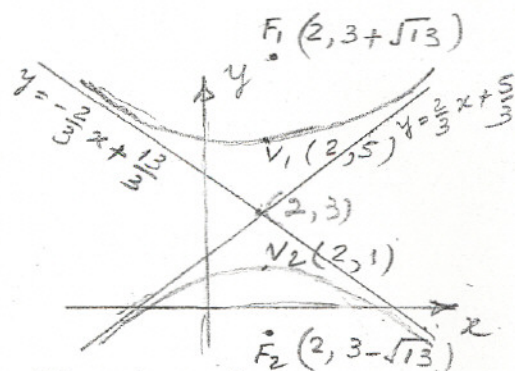
Foci: $(2, 3 + \sqrt{13})$, $(2, 3 - \sqrt{13})$

Asymptotes: $y - k = \frac{a}{b}(x - h)$

$$y - 3 = \frac{2}{3}(x - 2); \quad y - 3 = \frac{2x - 4}{3}$$

$$y = \frac{2}{3}x + \frac{5}{3}$$

$$\text{and } y = -\frac{2}{3}x + \frac{13}{3}$$



Eccentricity of hyperbola

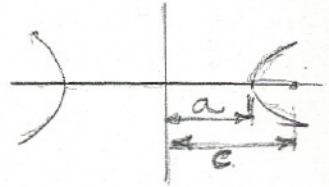
The eccentricity of hyperbolas determines its 'wideness'.

The eccentricity e of a hyperbola is the ratio of c to a where c is the distance from the center to focus and a is the length from the center to the vertex.

$$e = \frac{c}{a}$$

For hyperbola, $c > a$; therefore $e > 1$

As e increases, the graph becomes wider



Example: Find the equation of hyperbola, given eccentricity.

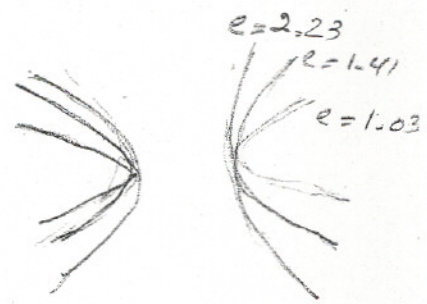
$e = \frac{3}{2}$ and focus is $(6, 0)$

$e = \frac{c}{a} = \frac{3}{2}$ $c = 6$

$\frac{6}{a} = \frac{3}{2}$ and $a = 4$

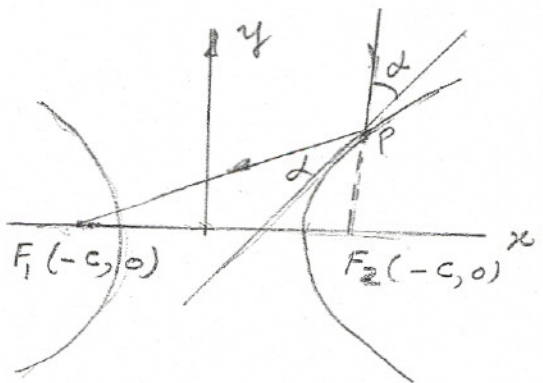
$c^2 = a^2 + b^2 = 36 = 16 + b^2$
 $b^2 = 20, b = \pm 2\sqrt{5}$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$



Application of Hyperbolas

A ray of light directed towards one focus of hyperbolic mirror is reflected towards the other focus.



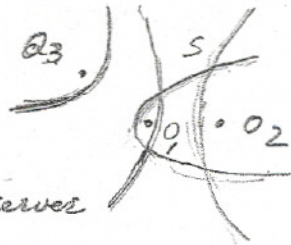
Suppose that a gun is fired from unknown source S .
 An observer at O_1 hears the sound of gun shot 1 second
 after another observer at O_2 .

Sound travels at a speed of $1100'$ /sec

It means that S is $1100'$ away from

O_2 than O_1 , because $d(S, O_1) - d(S, O_2)$

is constant and equal 1100 . The observer



O_3 hears the same sound two seconds later than O_1 , then

S will lie on the second hyperbola with foci O_1 and O_3 . The
 intersection of the two hyperbolas will locate the location
 of S .

Example: Two microphones, 1 mile apart record an
 explosion. Microphone B receives the sound 2 sec. before
 microphone A. Find the location of explosion.

As sound travels at $1100'$ /sec. distance $(d_2 - d_1) = 2a = 2200'$

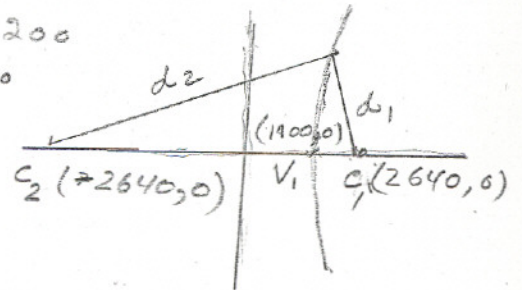
$$2c = 5280$$

$$c = 2640'$$

$$d_2 - d_1 = 2a = 2200$$

$$a = 1100$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$c^2 = a^2 + b^2$$

$$2640^2 = 1100^2 + b^2$$

$$b^2 = 2640^2 - 1100^2$$

$$b^2 = 5,759,600, \quad b = 2400$$

$$\frac{x^2}{1100^2} - \frac{y^2}{2400^2} = 1$$

If you have received sound at a third position C ,
 then you have to determine two other parabolas. The
 exact location is the point at which three parabolas
 intersect.