Section 9.3

Slope and Tangent Line:

The parametric curve $x = f(t)$ and $y = g(t)$ is called smooth if the derivatives $f'(t)$ and $g'(t)$ are continuous and never simultaneously zero. The slope dy/dx at (x, y) is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}, \quad f'(t) \neq 0$$

The tangent line is vertical at any point where $f'(t) = 0$ but $g'(t) \neq 0$.

Example 1. Find $\frac{dy}{dx}$ for the curve given by $x = \sin t$, $y = \cos t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)}{dx/dt}$$

Finding slope and concavity.

Example 2: $x = \sqrt{t}$, $y = \frac{1}{4}(t^2 - 4)$ $\quad t \geq 0$

Find the slope and the concavity:

$$\frac{dx}{dt} = \frac{1}{2}t^{-1/2} \qquad \frac{dy}{dt} = \frac{1}{4} \cdot 2t = \frac{t}{2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t/2}{t^{-1/2}/2} = t \cdot t^{1/2} = t^{3/2}$$

To find second derivative;

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dt/dx} = \frac{\frac{d}{dt}(t^{3/2})}{\frac{t^{-1/2}}{2}} = \frac{\frac{3}{2}t^{1/2}}{\frac{t^{-1/2}}{2}} = 3t$$

@ $(2, 3)$, $2 = \sqrt{t}$ & $t = 4$

$$\left.\frac{d^2y}{dx^2}\right|_{t=4} = 3 \times 4 = 12 > 0 \Rightarrow \text{graph is concave up}$$

Example 3: Find dy/dx and $\frac{d^2y}{dx^2}$ for the cycloid

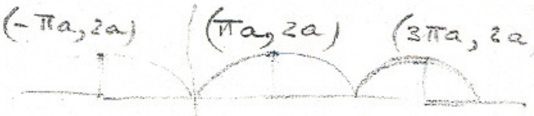
$$x = a(t - \sin t) \qquad y = a(1 - \cos t)$$

$$\frac{dx}{dt} = a(1 - \cos t) \qquad \frac{dy}{dt} = a \sin t.$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}.$$

@ $t = \pi, 3\pi \ldots$ (i.e., odd multiple of π)

$$\frac{dy}{dx} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1+1} = \frac{0}{2} = 0$$



The end points of the arches occur at even multiples of π, where both numerator and denominator are zero. These point are called cusps.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{\sin t}{1 - \cos t}\right)}{a(1 - \cos t)} = \frac{\frac{(1-\cos t)\cos t - \sin t(-\sin t)}{(1-\cos t)^2}}{a(1 - \cos t)}$$

$$= -\frac{1}{a(1-\cos t)^2}$$

Since the second derivative is negative for all negative multiples of π, $\Rightarrow \frac{d^2y}{dx^2} < 0$, The cyloid concaves downward.

<u>Note</u>: $\lim\limits_{t \to 2n\pi} \frac{dy}{dx} = \lim\limits_{t \to 2n\pi} \frac{\sin t}{1 - \cos t} = \lim\limits_{t \to 2n\pi} \frac{\cos t}{\sin t} = \pm \infty$

$\Rightarrow$ tangent line is a vertical line at the cusp point.

Example 4 A curve with two tangent lines at a point.

$$x = 2t - \pi \sin t \qquad y = 2 - \pi \cos t$$

crosses at $(0, 2)$. Find the equation of both tangent lines.

$t = \pm \pi/2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi \sin t}{2 - \pi \cos t} =$$

@ $t = \pi/2$, $\frac{dy}{dx} = \frac{\pi}{2}$.

@ $t = -\pi/2$, $\frac{dy}{dx} = -\pi/2$

Eq. of line: $y - 2 = \frac{\pi}{2} x$

" " " "

$x = 4y^2 - 4$

represent a parabola with a horizontal axis and vertex at $(-4, 0)$. the domain of the rectangular equation must be altered so that the graph matches the graph of the parametric equations.

Adjusting Domain After Eliminating the Parameter

$x = \dfrac{1}{\sqrt{t+1}} \qquad y = \dfrac{t}{t+1} \qquad t > -1$

$x^2 = \dfrac{1}{t+1}, \qquad t = \dfrac{1-x^2}{x^2}$

$y = \dfrac{(1-x^2)/x^2}{x^2} = 1 - x^2$

$y = 1 - x^2$

the rectangular equation $y = 1 - x^2$ is defined for all values of x. But the parametric equation is defined when $t > -1$ $\Rightarrow$ the domain of x must be restricted to positive values

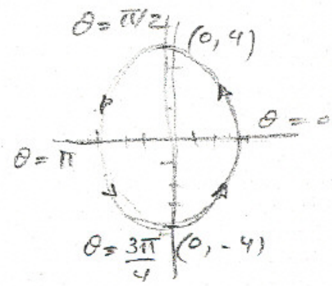
Using Trigonometric to Eliminate a Parameter

$x = 3\sin\theta \qquad y = 4\cos\theta \qquad 0 \leq \theta \leq 2\pi$

$\sin\theta = \dfrac{x}{3} \qquad \dfrac{y}{4} = \cos\theta$

$\dfrac{y}{4} = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \dfrac{x^2}{9}}$

$\dfrac{y^2}{16} = 1 - \dfrac{x^2}{9}$ or $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$



the graph is an ellipse, centered at $(0, 0)$, with vertices $(0, 4)$ and $(0, -4)$ and minor axis of length 3. Orientation is counter clockwise.

NOTE (a) Parametric equations

$x = h + a\cos\theta, \quad y = k + b\sin\theta \qquad 0 \leq \theta \leq 2\pi$

Ellipse is traced counter clockwise, $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

(b) Parametric equations $x = h + a\sin\theta, \ y = k + b\cos\theta$

Ellipse is traced clockwise. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Parametric equations tell the position, direction and speed at a given time

Find Parametric Equations.

Determination of parametric equation for a graph is not unique. You may find more than one different parametric representation for a given graph.

Example. Finding Parametric Equations for a given graph.

$y = 1 - x^2$

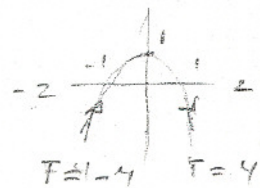
Find the parametric equations representing the graph of $y = 1 - x^2$, using following parameters.

(a) $t = x$ and (b) $m = \frac{dy}{dx}$ @ (x, y)

(a) $x = t$ and $y = 1 - t^2$

The parametric equations are $x = t$ and $y = 1 - t^2$

The orientation is from left to right.



(b) Express x and y in terms of parameter m

$m = \frac{dy}{dx} = -2x$

$x = -m/2$

$y = 1 - x^2 = 1 - \frac{m^2}{4}$

NOTE: The curve is from right to left.

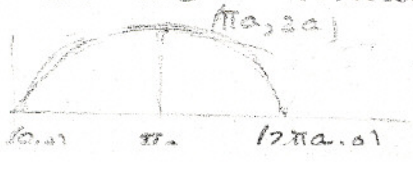
Parametric Equation of a Cycloid.

The use of parametric equations $x = x(t)$ and $y = y(t)$ is most advantageous when elimination of parameter t is impossible or would lead to an equation $y = f(x)$ that is more complicated than the original parametric equation.

The curve traced by a point P on the circumference of a circle of radius a rolling along a straight line is called a cycloid.

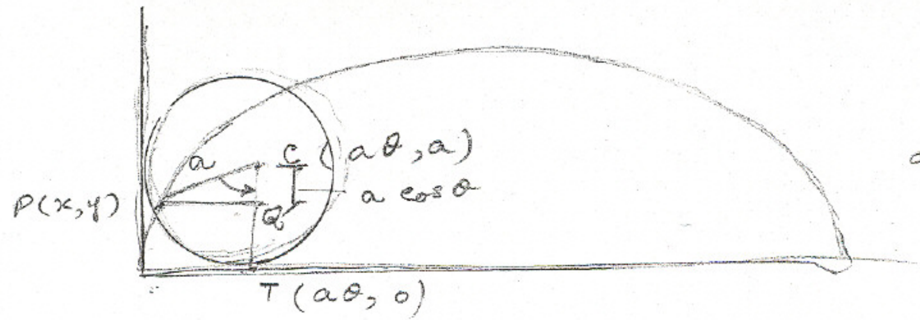
Let θ be the parameter to measure circular rotation.

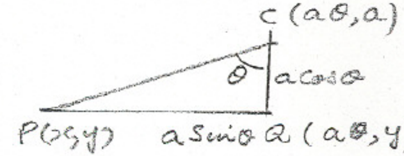
and Let $P(x, y)$ begin at the origin. When $\theta = 0$; P is at the origin. When $\theta = \pi$, P is at the maximum point $(\pi a, 2a)$.



When $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$

The distance the circle has rolled is $|OT|$ that is equal to to the length of the circumference subtended by the angle TCP.





$\overline{CQ} = a\cos\theta$, $\overline{PQ} = a\sin\theta$

$\overline{OT} = a\theta$, a is the radius of the circle.

From the right angle triangle PCQ

$a\theta - x = a\sin\theta$, $a - y = a\cos\theta$

or $x = a(\theta - \sin\theta)$

$y = a(1 - \cos\theta)$

The cycloid has sharp corners: @ $x = 2\pi n a$

$x' = a(1 - \cos\theta)$, $x'(2\pi n) = a - a\cos 2\pi n = 0$

$y' = a\sin\theta$, $y'(2\pi n) = a\sin 2\pi n = 0$

Between these two points, the cycloid is smooth.