14.3: Conservative Vector Fields and Independence of Path

1. In section 14.2 we pointed out that in a gravitational field the work done by gravity on an object moving between two points is independent of the path taken by the object. In this section, we will study the important generalization of this result – called the **Fundamental Theorem of Line Integrals.**

2. Theorem 14.5: Fundamental Theorem of Line Integrals

Let C be a piecewise smooth curve lying in an open region R given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad , \ a \le t \le b$$

If F(x,y) = Mi + Nj is conservative in R, and M and N are continuous in R then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

Where f is a potential function of **F**. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.

Notice how the Fundamental Theorem of Line Integrals is similar to the Fundamental Theorem of Calculus in section 4.4

The Fundamental Theorem of Line *Integrals states that if the vector field F is conservative, then the line integral between any 2 points is simply the difference in the values of the potential function f at these points.*

3. A region in the plane or space is **connected** if any two points in the region can be joined by a piecewise smooth curve lying entirely within the region. .

Theorem 14.6: Independence of Path and Conservative Vector Fields

If F is continuous on an open region, then the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ is independent of path if and only if **F** is conservative.

4. **Theorem 14.7: Equivalent Conditions**

Let **F**(x,y,z) = M**i** + N**j** + P**k** have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent.

- **F** is conservative. That is $\mathbf{F} = \nabla f$ for some function f.
- $\int_C \mathbf{F} \bullet d\mathbf{r}$ is independent of path.
- $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed curve C in R.

Theorem 14.7 give you options for evaluating a line integral involving a conservative vector field. You can use a potential function, or it might be more convenient to choose a particularly simple that, such as a straight line.

5. **Conservation of Energy – One of the most important laws of physics**

The Law of Conservation of Energy states in a conservative field, the sum of the potential and kinetic energies of an object remains constant from point to point. You can use the Fundamental Theorem of Line Integrals to derive this law. See p 1038.