14.2: Line Integrals

1. **Piecewise Smooth Curves:** A curve C is piecewise smooth if the interval [a,b] can be partitioned into a finite number of subintervals, on each of which C is smooth.

2. **Definition of Line Integral**

   If f is defined in a region containing a smooth curve C of finite length, then the **line integral of f along C** is given by

$$\int_C f(x,y)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i \qquad \text{(Plane)}$$

OR

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i \qquad \text{(Space)}$$

Provided the limit exists.

   Evaluation of a line integral is best accomplished by converting to a definite integral.

3. **Theorem 14.4: Evaluation of a Line Integral as a Definite Integral**

   Let f be continuous in a region containing a smooth curve C. If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where $a \le t \le b$, then

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

   If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \le t \le b$ then

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

   Note: $\int_C 1\,ds = \int_a^b \|\mathbf{r}'(t)\|\,dt$ = length of curve C

   For the parameterization given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, it is helpful to remember the form

ds as $ds = \|\mathbf{r}'(t)\|\,dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,ds$

4. Definition of Line Integral of a Vector Field

Let F be a continuous vector field defined on a smooth curve C given by **r**(t), $a \le t \le b$. The **line integral of F** on C is given by

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet \mathbf{T} ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \bullet \mathbf{r}'(t) dt$$

Application: The total work done is given by the following integral:

$$\Delta W_i = \text{(force)(distance)}$$

$$W = \int_C \mathbf{F}(x, y, z) \bullet \mathbf{T}(x, y, z) ds$$

5. **For line integrals of vector functions, the orientation of the curve C is important**. If the orientation of the curve is reversed, the unit tangent vector **T**(t) is changed to –**T**(t), and you obtain

$$\int_{-C} \mathbf{F} \bullet d\mathbf{r} = -\int_C \mathbf{F} \bullet d\mathbf{r}$$

6. Line Integrals in Differential Form

A second commonly used form of line integrals is derived from the vector field notation used in the preceding section. If F is vector field of form **F**(x,y) = M**i** + N**j,** and C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F} \bullet d\mathbf{r}$ is often written M dx + N dy.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C F \bullet \frac{d\mathbf{r}}{dt} dt = \int_C (Mdx + Ndy)$$

This differential form can be extended to three variables.