14.1: Vector Fields

1. Functions that assign a vector to a point in the plane or a point in space are called **vector fields**. They are useful is representing various types of **force fields** and **velocity fields**.

2. **Definition of a Vector Field**

Let M and N be functions of two variables x and y, defined on a plane region R. The function F defined by

$$\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j} \quad \text{(Plane)}$$

is called a **vector field over R.**

Let M, N, and P be functions of three variables x, y, and z defined on a solid region Q. The function F defined by

$$\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \quad \text{(Space)}$$

is called a **vector field over Q.**

A vector field is continuous at a point if each of its component functions M, N, and P are continuous at that point.

From this definition, you can see that the gradient is one example of a vector field.

3. **Some common physical examples of vector fields are velocity fields, gravitational fields, and electrical fields**

- Gravitational fields are defined by Newton's Law of Gravitation, which states that the force of attraction exerted on a particle of mass m_1 located at (x,y,z) by a particle of mass m_2 located at (0,0,0) is given by

$$\mathbf{F}(x,y,z) = \frac{-Gm_1m_2}{x^2+y^2+z^2}\,\mathbf{u}$$

where G is the gravitational constant and u is the unit vector in the direction to (x,y,z)

- Electronic force fields are defined by Coulomb's Law, which states that the force exerted on a particle with electric charge q_1 located at (x,y,z) by a particle with electric charge q_2 located at (0,0,0) is given by

$$\mathbf{F}(x,y,z) = \frac{cq_1q_2}{\|\mathbf{r}\|^2}\,\mathbf{u}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$, and c is a constant

4. Definition of Inverse Square Field

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a position vector. The vector field **F** is an **inverse square** if

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}$$

where k is a real number and $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$ is a unit vector in the direction of **r**.

5. Definition of Conservative Vector Field

A vector field F is called conservative if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the **potential function** of **F**.

6. Theorem 14.1: Test for Conservative Vector Field in the Plane

Let M and N have continuous first partial derivatives on an open disk R. The vector field given by **F**(x,y) = M**i** + N**j** is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

*Notice, this theorem does not tell you how to find **F**.*

7. Definition of Curl of a Vector Field

The curl of **F**(x,y,z) = M**i** + N**j** + P**k** is

$$\textbf{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial m}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

*Note: if **curl F = 0**, we say **F is irrotational***

8. Theorem 14.2: Test for Conservative Vector Field in Space

Suppose that M, N, and P have continuous first partial derivatives in an open sphere Q in space. The vector field given by **F**(x,y,z) = M**i** + N**j** + P**k** is conservative if and only if

curl F(x,y,z) **= 0**

That is, F is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial m}{\partial z}, \text{ and } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

9. Definition of Divergence of a Vector Field: Divergence measures the rate of particle flow per unit volume at a point.

The divergence of **F**(x,y) = M**i** + N**j** is

$$\text{div } \mathbf{F}(x, y) = \nabla \bullet \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \qquad \text{(Plane)}$$

The divergence of **F**(x,y) = M**i** + N**j + P****k** is

$$\text{div } \mathbf{F}(x, y, z) = \nabla \bullet \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \qquad \text{(Space)}$$

*If div **F** = 0, then **F** is said to be **divergence free.***

10. Theorem 14.3: Relationship Between Divergence and Curl

If of **F**(x,y) = M**i** + N**j + P****k** is a vector field and M, N, and P have continuous second partial derivatives, then

div **(curl F) =** 0