13.7: Triple Integrals in Cylindrical and Spherical Coordinates

1. Triple Integrals in Cylindrical Coordinates

Recall from 10.7 that the rectangular conversion equations for cylindrical coordinates are

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$
$$z = z$$

If R is simple, the iterated form of the triple integral in cylindrical form is

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

To visualize a particular order of integration, it helps to view the iterated integral in terms of three sweeping motions – each adding another dimension to the sold.

2. Triple Integrals in Spherical Coordinates

Triple integrals involving spheres or cones are often easier to evaluate by converting to spherical coordinates. Recall the conversion formulas

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan\theta = \frac{y}{x}$$
$$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

A triple integral in spherical coordinate can take the form

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

This formula can be modified for different orders of integration and generalized to include regions with variable boundaries.