13.6: Triple Integrals and Applications

1. Definition of Triple Integral

If f is continuous over a bounded solid region Q, then the **triple integral** of f over Q is defined as

$$\iiint_Q f(x,y,z)\,dV = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

Provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of Q} = \iiint_Q dV$$

2. **Theorem 13.4: Evaluation by Iterated Integrals** (version of Fubini's Theorem)

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x,y) \le z \le g_2(x,y)$$

Where h_1, h_2, g_1, and g_2 are continuous functions. Then

$$\iiint_Q f(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dzdydz$$

To evaluate a triple iterated integral in the order dz dy dz, hold both x and y constant for the inner most integration. Then hold x constant for the second integration. For other orders, you can follow a similar procedure.

To find the limits for a particular order of integration, it is advisable to first determine the inner most limits, which may be functions of the outer tow variables. Then by projecting the solid Q onto the coordinate plane of the outer two, you can determine their limits of integration like you did for double integration.

Some orders of integration will be easier than others.

3. **Center of Mass and Moments of Inertia**

Consider a solid region Q whose density at (x,y,z) is given by the density function ρ. The center of mass of a solid region Q of mass m is given by $\left(\bar{x},\bar{y},\bar{z}\right)$, where

Mass of the solid: $m=\iint_Q\int \rho(x,y,z)dV$

First Moment about yz-plan: $M_{yz}=\iint_Q\int x\rho(x,y,z)dV$

First Moment about xz-plane: $M_{xz}=\iint_Q\int y\rho(x,y,z)dV$

First Moment about xy-plane: $M_{xy}=\iiint_Q z\rho(x,y,z)dV$

And

$$\left(\bar{x},\bar{y},\bar{z}\right)=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)$$

The second moment (or moments of inertia) about the x, y, and z axes are as follows

Moment of inertia about x axis: $I_x=\iiint_Q (y^2+z^2)\rho(x,y,z)dV$

Moment of inertia about y-axis: $I_y=\iiint_Q (x^2+z^2)\rho(x,y,z)dV$

Moment of inertia about z-axis: $I_z=\iiint_Q (x^2+y^2)\rho(x,y,z)dV$

For problems requiring the calculations of all three moments, you can apply the additive property of triple integrals and writing

$$I_x=I_{xz}+I_{xy}$$

$$I_y=I_{yz}+I_{xy}$$

$$I_z=I_{yz}+I_{xz}$$

Where

$$I_{xy}=\iiint_Q z^2\rho(x,y,z)dV$$

$$I_{xz}=\iiint_Q y^2\rho(x,y,z)dV$$

$$I_{yz}=\iiint_Q x^2\rho(x,y,z)dV$$

4. **In engineering and physics, the moment of inertia of a mass is used to find the time required for a mass to reach a given speed of rotation about an axis. The greater the moment of inertia, the longer a force must be applied for the mass to reach the given speed.**