13.4: Center of Mass and Moments of Intertia

1. Definition of Mass of a Planar Lamina of Variable Density

If ρ is a continuous density function on the lamina corresponding to a plane region R, then the mass m of the lamina is given by

$$m = \int_R \int \rho(x, y)\,dA \quad \text{Variable Density}$$

2. Moments and Center of Mass of a Variable Density Planar Lamina

Let ρ be a continuous density function on the planar lamina R. The moments of mass with respect to the x and y axes are

$$M_x = \int_R \int y\rho(x, y)\,dA \quad \text{and} \quad M_y = \int_R \int x\rho(x, y)\,dA$$

If m is the mass of the lamina, then the center of mass is

$$\left(\bar{x}, \bar{y}\right) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

If R represents a simple plane region rather than a lamina, the point $\left(\bar{x}, \bar{y}\right)$ is called the **centroid** of the region.

- For some planar laminas, you can determine the center of mass using symmetry rather than using integration.
- Although you can think of the moments as measuring the tendency to rotate about the x and y axis, the calculation of moments is usually an intermediate step toward finding the center of mass.
- The moments of mass used to determine the center of mass of a lamina are sometimes called the **first moments** about the x and y axis.

3. The Second Moment or Moment of Inertia

In the same way that mass is a measure of the tendency to resist a change in a straight line motion, the moment of inertia about a line is a measure of the tendency of matter to resist change in a rotational motion.

The second moments denoted I_x and I_y are the product of mass time the square of the distance.

$$I_x = \int_R \int y^2 \rho(x,y)\,dA \quad \text{and} \quad I_y = \int_R \int x^2 \rho(x,y)\,dA$$

The sum of the moments I_x and I_y is called the polar moment of inertia and is denoted I_0.

The moment of inertia of a revolving lamina can be used to measure its kinetic energy.

$$E = \frac{1}{2} I \omega^2$$

Kinetic energy of a mass moving in a straight line is proportional to its mass, but kinetic energy of a mass revolving about an axis is proportional to its moment of inertia.

4. The radius of gyration $\overset{=}{r}$ of a revolving mass m with moment of inertia I is defined to be

$$\overset{=}{r} = \sqrt{\frac{I}{m}}$$

The radius of gyration can also be calculated with respect to the x and y axes. See page 968