12.8: Extrema of Functions of Two Variables

1. **Theorem 12.15: Extreme Value Theorem**

   Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

   - There is at least one point in R where f takes on a minimum value.
   - There is at least one point in R where f takes on a maximum value.

   Recall: A minimum is also called an absolute minimum and a maximum is also called an absolute maximum.

2. **Definition of Relative Extrema**

   Let f be a function defined on a region R containing (x_0,y_0).

   - The function f has a **relative minimum** at (x_0,y_0) if
     $$f(x,y) \geq f(x_0,y_0)$$
     for all (x,y) in an open disc containing (x_0,y_0).

   - The function f has a **relative maximum** at (x_0,y_0) if
     $$f(x,y) \leq f(x_0,y_0)$$
     for all (x,y) in an open disc containing (x_0,y_0).

3. **Definition of Critical Point**

   Let f be a function defined on a region R containing (x_0,y_0). The point (x_0,y_0) is a critical point of f if one of the following is true.

   - $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$
   - $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ does not exist.

4. **Theorem 12.16: Relative Extrema Occur Only at Critical Points.**

5. Theorem 12.17: Second Partials Test

Let f have continuous second partial derivatives on an open region containing a point (a,b) for which

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$$

- If d > 0 and $f_{xx}(a,b) > 0$, then f has a **relative minimum** at (a,b).
- If d > 0 and $f_{xx}(a,b) < 0$, then f has a **relative maximum** at (a,b).
- If d < 0 then (a, b, f(a,b)) is a **saddle point**.
- The test is **inconclusive if d = 0**.