12.7: Tangent Planes and Normal Lines

1. So far we have represented surfaces in space primarily by equations of the form z = f(x,y). In some cases it is more convenient to use the more general form F(x,y,z) = 0. You can convert to the general form by defining F as

$$F(x,y,z) = f(x,y) - z$$

Since f(x,y) – z = 0, you can consider S to be the level surface of F given by F(x,y,z)=0.

2. **Definition of Tangent Plane and Normal Plane**

Let F be differentiable at the point $P(x_0,y_0,z_0)$ on the surface S given by F(x,y,z) = 0 such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$

- The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S at P.**
- The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S at P.**

Recall: The equation of a normal line is $\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}$ or

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

3. **To find a unit normal vector to a surface**, you can normalize the gradient vector.

- $\mathbf{n} = \dfrac{\nabla F}{\|\nabla F\|}$

4. **Theorem 12.13: Equation of Tangent Plane**

Let F be differentiable at (x_0,y_0,z_0), then an equation of the tangent plane to the surface given by S by F(x,y,z) = 0 at (x_0,y_0,z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Because $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane at (x_0, y_0, z_0), it must be orthogonal to every vector in the tangent plane, and you have
$\nabla F(x_0, y_0, z_0) \bullet \mathbf{v} = 0$

5. The Angle of Inclination of a Plane

The angle of inclination of a plane is defined to be the angle θ, $0 \le \theta \le \pi/2$, between the given plane and the xy-plane. Since vector **k** is normal to the xy-plane, you can use the formula for the cosine of the angle between two planes.

$$\cos\theta = \frac{|\mathbf{n} \bullet \mathbf{k}|}{\|\mathbf{n}\|\|\mathbf{k}\|} = \frac{|\mathbf{n} \bullet \mathbf{k}|}{\|\mathbf{n}\|}$$

6. Theorem 12.14: Gradient is Normal to Level Surfaces

If F is differentiable at (x_0,y_0,z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0,y_0,z_0)