12.6: Directional Derivatives and Gradients

1. Definition of Directional Derivative

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then the directional **derivative of f in the direction of u**, denoted by $D_{\mathbf{u}}f$ is

$$D_{\mathbf{u}} f(x,y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x,y)}{t}$$

provided the limit exists.

2. Theorem 12.9: Directional Derivative

If f is a differentiable function of x and y, then the directional derivatives of f in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

There are infinitely many directional derivatives to a surface at a given point – one for each direction specified by **u**. Two of these are the partial derivatives f_x and f_y.

i) Direction of positive x-axis ($\theta = 0$): $\mathbf{u} = \cos 0\mathbf{i} + \sin 0\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}} f(x,y) = f_x(x,y)\cos 0 + f_y(x,y)\sin 0 = f_x(x,y)$$

ii) Direction of positive y-axis ($\theta = \pi/2$): $\mathbf{u} = \cos(\pi/2)\,\mathbf{i} + \sin(\pi/2)\,\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}} f(x,y) = f_x(x,y)\cos\frac{\pi}{2} + f_y(x,y)\sin\frac{\pi}{2} = f_y(x,y)$$

3. **The gradient** of a function of two variables is a vector valued function of two variables. This function has many important uses, which we will discover.

Definition of Gradient of a Function of Two Variables

Let z = f(x,y) be a function of x and y such that f_x and f_y exist. Then the gradient of f, denoted $\nabla f(x,y)$ is the vector

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

We read $\nabla f(x,y)$ as "del f". Another notation for the gradient is **grad** f(x,y). $\nabla f(x,y)$ is a vector in the plane, not a vector in space.

4. **Theorem 12.10: Alternative Form of the Directional Derivative**

   If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector u is

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \bullet \mathbf{u}$$

5. **Applications of the Gradient**

   In many applications we would like to know in which direction to move so that f(x,y) increases most rapidly. This direction is called the direction of steepest ascent, and it is given by the gradient.

Theorem 12.11: Properties of the Gradient

- If $\nabla f(x,y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x,y) = 0$ for all **u**.
- The direction of maximum increase of f is given by $\nabla f(x,y)$. The maximum value of $D_{\mathbf{u}} f(x,y)$ is $\|\nabla f(x,y)\|$.
- The direction of minimum increase of f is given by $-\nabla f(x,y)$. The maximum value of $D_{\mathbf{u}} f(x,y)$ is $-\|\nabla f(x,y)\|$.

Imagine a skier skiing down a mountainside. $-\nabla f(x,y)$ indicates the compass direction the skier should take to ski the path of steepest descent.

On a hot metal plate, $\nabla f(x,y)$ gives the direction of greatest temperature increase.

Just remember, the gradient will change as soon as you move from a given point.

6. **Theorem 12.12: Gradient is Normal to Level Curves**

If f is differentiable at (x_0,y_0) and $\nabla f(x_0,y_0)\neq \mathbf{0}$, then $\nabla f(x_0,y_0)$ is normal to the level curve at (x_0,y_0).

7. Directional Derivative and Gradient for Three Variables

If f is a differentiable function of x, y, and z with continuous first partial derivatives. The **directional derivatives of f** in the direction of the unit vector **u** = a**i** + b**j** + c**k** is given by

$$D_{\mathbf{u}} f(x,y,z) = af_x(x,y,z) + bf_y(x,y,z) + cf_z(x,y,z)$$

The **gradient of f** is denoted to be

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

Properties of the gradient are as follows.

- $D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \bullet \mathbf{u}$
- If $\nabla f(x,y,z) = \mathbf{0}$**,** then $D_{\mathbf{u}} f(x,y,z) = 0$ for all **u**.

- The direction of maximum increase of f is given by $\nabla f(x,y,z)$. The maximum value of $D_{\mathbf{u}} f(x,y)$ is $\|\nabla f(x,y,z)\|$.

- The direction of minimum increase of f is given by $-\nabla f(x,y,z)$. The maximum value of $D_{\mathbf{u}} f(x,y)$ is $-\|\nabla f(x,y,z)\|$.