12.4: Differentials

1. Definition of Total Differential

If x = f(x, y) and Δx and Δy are increments of x and y, then the differentials of the independent variables x and y are

$$dx = \Delta x \text{ and } dy = \Delta y$$

are the total differential of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x,y)dx + f_y(x,y)dy$$

This definition can be extended to functions of three or more variables.

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial u}du$$

2. Definition of Differentiable

A function f given by z = f(x,y) is **differentiable** at (x_0,y_0) if Δz can be expressed in the form

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

Where both ε_1 and $\varepsilon_2 \to 0$ as (Δx,Δy) approach (0,0). The function f is **differentiable in a region R** if it is differentiable at each point in R.

For small Δx and Δy, you can use the approximation Δz ≈ dz.

Notice that for a function of two variable, the existence of the partial derivatives does not guarantee that the function is differentiable. See the next theorem 12.4

3. Theorem 12.4: Sufficient Condition for Differentiability

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

4. Approximation by Differentials

For small Δx and Δy, you can use the approximation $\Delta z \approx dz$. Recall the partial derivatives can be interpreted as the slopes of the surface in the x and y directions. That means

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial x}\Delta y$$

Represents the change in the height of a plane that is tangent to the surface at the point (x,y,f(x,y)). Because a plane in space is represented by a linear equation in the variables x, y, and z, the approximation of Δz by dz is called a **linear approximation.**

5. A **function of three variables w = f(x,y,z) is called differentiable** at (x,y,z) provided that

$$\Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

Where both ε_1 and ε_2 and $\varepsilon_3 \to 0$ as (Δx,Δy, ,Δz) approach (0,0). If f is a function of x, y, and z where f_x, f_y , f_z are continuous in an open region R, then f is differentiable on R.

6. Just like in Chapter 3, you can use differentials to **approximate propagated error and relative error.**

7. Theorem 12.5: Differentiability Implies Continuity.