12.3: Partial Derivatives

1. Definition of Partial Derivatives of a Function of Two Variables

If z = f(x, y), then the first partial derivatives of f with respect of x and y are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$
$$f_y(x,y) = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Provided the limit exists

- **To find f_x you consider y constant and differentiate with respect to x.**
- **To find f_y, you consider x constant and differentiate with respect to y.**

2. Notation for First Partial Derivatives

For z = f(x, y), the first partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = z_x = \frac{\partial z}{\partial x}$$

And

$$\frac{\partial}{\partial y} f(x,y) = f_y(x,y) = z_y = \frac{\partial z}{\partial y}$$

The first partial evaluated at the point (a,b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a,b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

3. Informally, we say that the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (x_0,y_0,z_0) denotes the slopes of the surface in the x and y directions.

4. Partial Derivatives of a Function of Three or More Variables

The concept of a partial derivative can be extended to functions of three or more variables. For instance w = f(x, y, z), there are three partial derivatives, each formed by holding two of the variables constant. That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x.

5. Higher Order Partial Derivatives

It is possible to take second, third, and higher partial derivatives, provided they exist.

- Differentiate twice with respect to x: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}$

- Differentiate twice with respect to y: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$

- Differentiate first with respect to x and then with respect to y: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y \partial x}=f_{xy}$

- Differentiate first with respect to y and then with respect to x: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x \partial y}=f_{yx}$

The last two are called **mixed partial derivatives.**

6. Theorem 12.3: Equality of Mixed Partial Derivatives

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y)$$