12.2 Notes: Limits and Continuity

1. Limit of a Function of Two Variables

Let f be a function of two variables defined, except possibly at (x_0,y_0), on an open disk centered at (x_0,y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that

$$\left| f(x,y) - L \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

This definition is similar to the definition of a function of a single variable. However, there is a critical difference. To determine where a limit exists for a function of a single variable, you need only test the approach from the left and right. For a function of two variables the statement $(x,y)\to(x_0,y_0)$ **means that (x, y) is allowed to approach (x_0, y_0) from any direction**. **If the value of the limit is not the same for all possible approaches, or paths, the limit does not exist.**

2. Definition of Continuity of a Function of Two Variables

A function f of two variables is **continuous at the point (x_0,y_0)** in an open region R if f(x_0,y_0) is defined and equal to the limit of f(x, y) as (x, y) approaches (x_0,y_0). That is

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

If a function is not continuous, there are two types of discontinuity: **removable** and **nonremoveable** . If you can redefine f at a singe point to make the function continuous, the discontinuity is considered removable.

3. Theorem 12.1: Continuous Functions of Two Variables

If k is a real number and f and g are continuous at (x_0,y_0), then the following functions are continuous at (x_0,y_0).

Scalar multiple: kf

Sum and Difference: f ± g

Product: fg

Quotient: f/g , if g does not equal 0

4. Theorem 12.2: Continuity of a Composite Function

If h is continuous at (x_0,y_0) and g is continuous at $h(x_0,y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0,y_0). This is

$$\lim_{(x,y)\to(x_0,y_0)} g(h(x,y) = g(h(x_0,y_0))$$

5. Definition of Continuity of a Function of Three Variables
A function f of three variables is continuous at a point (x_0,y_0,z_0) in an open region R if $f(x_0,y_0,z_0)$ is defined and equal to the limit of f(x, y, z) as (x, y, z) approaches (x_0,y_0,z_0). That is

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$$

The function f is **continuous in the open region R** if it is continuous at every point in R.