11.4: Tangent Vectors and Normal Vectors

A. Definition of Unit Tangent Vector

Let C be a smooth curve represented by **r** on an open interval I. The **unit tangent vector T**(t) at t is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad \text{, where } \mathbf{r}'(t) \neq \mathbf{0}$$

Recall "smoothness" is sufficient to guarantee that a curve has a unit tangent vector.

B. The **tangent line to a curve at a point** is the line passing through the point and parallel to the unit tangent vector.

C. Definition of Principal Unit Normal Vector

Let C be a smooth curve represented by **r** on an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then **the principal unit normal vector** at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

The **principal unit normal vector** can be difficult to evaluate algebraically. For plane curves, you can simplify the algebra by finding

$$\mathbf{T}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

and observing that **N**(t) must be either

$$\mathbf{N}_1(t) = y(t)\mathbf{i} - x(t)\mathbf{j} \quad \text{or} \quad \mathbf{N}_1(t) = -y(t)\mathbf{i} + x(t)\mathbf{j}$$

The principal unit normal vector **N** is the one that points toward the concave side of the curve. Recall that **T**(t) points in the direction the object is moving, whereas **N**(t) is orthogonal to **T**(t) and points in the direction the object is turning.

D. In general, part of the acceleration (the tangential component) acts in the line of motion and part (the normal part) acts perpendicular to the line of motion. To find these two components, you use the unit vectors **T**(t) and **N**(t), in much the same way as do **i** and **j** in representing vectors in space.

$$\mathbf{a}(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t)$$

Theorem 11.4: Acceleration Vector

If **r**(t) is the position vector for a smooth curve C and **N**(t) exists, then the acceleration vector a(t) lies in the plane determined by T(t) and **N**(t).

E. Theorem 11.5: Tangential and Normal Components of Acceleration

If **r**(t) is the position vector for a smooth curve C, for which **N**(t) exists, then the tangential and normal components of acceleration are as follows.

$$a_T = D_t\left[\|\mathbf{v}\|\right] = \mathbf{a}\bullet\mathbf{T} = \frac{v\bullet a}{\|\mathbf{v}\|}$$

$$a_N = \|\mathbf{v}\|\|\mathbf{T}'\| = \mathbf{a}\bullet\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_T{}^2}$$

Note that $a_N \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration.**