11.2: Differentiation and Integration of Vector-Valued Functions

A. Definition of the Derivative of a Vector-Valued Function

The derivative of a vector valued function r is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

For all t for which the limit exists. If r'(c) exists, the r is differentiable at c. If r'(c) exists for all c in an open interval I, then r is differentiable on the interval I. Differentiability of vector valued functions can be extended to closed intervals by considering one-sided limits.

B. Theorem 11.1: Differentiation of Vector Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t, then

$$\mathbf{r}(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

$$\mathbf{r}(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Higher order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

C. The parameterization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is **smooth on an open interval** I if f', g', and h' are continuous on I and $r'(t) \neq 0$ for any value of t in the interval I.

D. Theorem 11.2: Properties of the Derivative

$$D_t\left[c\mathbf{r}(t)\right] = c\mathbf{r}'(t)$$

$$D_t\left[\mathbf{r}(t) \pm \mathbf{u}(t)\right] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

$$D_t\left[f(t)\mathbf{r}(t)\right] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$$

$$D_t\left[\mathbf{r}(t) \bullet \mathbf{u}(t)\right] = \mathbf{r}(t) \bullet \mathbf{u}'(t) + \mathbf{r}'(t) \bullet \mathbf{u}(t)$$

$$D_t\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$$

$$D_t\left[\mathbf{r}(f(t))\right] = \mathbf{r}'(f(t))\, f'(t)$$

$$\text{If } \mathbf{r}(t) \bullet \mathbf{r}(t) = c, \text{ then } \mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$$

E. Definition of Integration of Vector Valued Functions

1. If **r**(t) = f(t)**i** + g(t)**j** , where f and g are continuous on [a,b], then **the indefinite integral (antiderivative) or r is**

$$\int \mathbf{r}(t)dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j}$$

And its **definite integral** over the interval [a,b] is

$$\int_a^b \mathbf{r}(t)dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j}$$

2. If **r**(t) = f'(t)**i** + g'(t)**j** + h'(t)**k,** where f, g, and h are continuous on [a,b] then the **indefinite integral (antiderivative) of r is**

$$\int \mathbf{r}(t)dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int h(t)dt\right]\mathbf{k}$$

And its **definite integral** over the interval [a,b] is

$$\int_a^b \mathbf{r}(t)dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j} + \left[\int_a^b h(t)dt\right]\mathbf{k}$$