10.5: Lines and Planes in Space

A. Theorem 10.11: Parametric Equations of a Line in Space

A line L parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the parametric equations

$$x = x_1 + at$$
$$y = y_1 + bt$$
$$z = z_1 + ct$$

The vector **v** is the **direction vector** for the line L, and a, b, and c are the **direction numbers**.

B. If the direction numbers a, b, and c are all nonzero, you can eliminate the parameter t to obtain the symmetric equations of a line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

C. Theorem 10.12: Standard Equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having a normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented, in **standard form**, by the equation

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

By regrouping terms, you can obtain the **general form** of the equation of a plane in space.

$$ax + by + cz + d = 0$$

D. If two distinct planes intersect, you can determine the angle $(0 \le \theta \le \pi/2)$ between them from the angle between their normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$. The angle between $\mathbf{n_1}$ and $\mathbf{n_2}$ is equal to the angle between the two planes.

$$\cos\theta = \frac{|\mathbf{n_1} \bullet \mathbf{n_2}|}{\|\mathbf{n_1}\|\|\mathbf{n_2}\|}$$

Two planes with normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ are

1. perpendicular if $\mathbf{n_1} \bullet \mathbf{n_2} = 0$
2. parallel if $\mathbf{n_1}$ is a scalar multiple of $\mathbf{n_2}$.

E. Sketching Planes in Space

To graph planes by hand, it is often helpful to make traces where the plane intersects each coordinate plane. Find each trace:

1. xy trace – set z = 0
2. yz trace – set x = 0
3. xz trace – set y = 0

If the equation of a plane has a missing variable, the plane must be parallel to the axis represented by the missing variable.

If two variable are missing, it is parallel to the coordinate plane represented by the missing variables.

F. Theorem 10.13: Distance Between a Point and a Plane

The distance between a plane and a point Q (not in the plane) is

$$D = \left\| proj_{\mathbf{n}} \overrightarrow{PQ} \right\| = \frac{\left| \overrightarrow{PQ} \bullet \mathbf{n} \right|}{\left\| \mathbf{n} \right\|}$$

Where P is a point in the plane and **n** is normal to the plane.

Alternative Distance Formula: The distance between a point $Q(x_0, y_0, z_0)$ and the plane
ax + by + cz + d = 0 is

$$D = \frac{\left| ax_0 + by_0 + cz_0 + d \right|}{\sqrt{a^2 + b^2 + c^2}}$$

G. Theorem 10.14: Distance Between a Point and a Line in Space

The distance between a point Q and a line in space is given by

$$D = \frac{\left\| \overrightarrow{PQ} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|}$$

Where **u** is the direction vector for the line and P is a point on the line