10.4: The Cross Product of Two Vectors in Space

A. Many applications if physics, engineering, and geometry involving find a vector in space that is orthogonal to two given vectors. The cross product of two vectors produces such a vector. Cross product is also called vector product.

Definition of Cross Product to Two Vectors in Space

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The **cross product of u and v** is a the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 = u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

A convenient way to calculate the cross product is to use the following determinant form.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

B. **Theorem 10.7: Algebraic Properties of the Cross Product** See Page 745

C. **Geometric Properties of the Cross Product**

Let **u** and **v** be nonzero vectors in space, and let θ be the angle between **u** and **v**.

1. **u** x **v** is orthogonal to both **u** and **v.**
2. ||**u** x **v**|| = ||**u**|| ||**v**|| sin θ
3. **u** x **v** = **0** if and only if u and v are scalar multiples of each other.
4. ||**u** x **v**|| = area of parallelogram having **u** and **v** as adjacent sides.

Note: The cross product **u** x **v** is perpendicular to each of **u** and **v**. It's direction is given by the thumb of the right hand when your fingers rotate from **u** to **v**.

D. In physics, the cross product can be used to measure **torque** – **the moment M of a force F about a point P**. If the point of application of the force is Q, the moment **F** about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F}$$

The magnitude of the moment **M** measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise (using the right-hand rule) about an axis directed along the vector **M**.

E. Theorem 10.9: The Triple Scalar Product

For **u** = u_1**i** + u_2**j** + u_3**k** and **v** = v_1**i** + v_2**j** + v_3**k** and **w** = w_1**i** + w_2**j** + w_3**k**, the triple scalar product is given by

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

F. Theorem 10.10: Geometric Property of Triple Scalar Product

The volume V of a parallelepiped with vectors u, v, and w as adjacent edges is given by

$$V = \left| \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) \right|$$