10.3: The Dot Product of Two Vectors

A. Definition of Dot Product

The dot product of $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$ is $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2$

The dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Note: The dot product of two vectors yields a scalar, it is also called the inner product or scalar product of two vectors.

B. Theorem 10.4: Properties of the Dot Product

1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
3. $c(\mathbf{u} \bullet \mathbf{v}) = c\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet c\mathbf{v}$
4. $\mathbf{0} \bullet \mathbf{v} = 0$
5. $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$

C. Theorem 10.5: Angle Between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

This can be rewritten as $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ to find the dot product if θ is know.

Note: The terms perpendicular and orthogonal and normal all mean essentially the same thing. However, we usually say two vectors are orthogonal, two lines are perpendicular, and a vector is normal to a given line or plan.

D. In space it is convenient to measure direction in terms of the angles between the nonzero vector **v** and the three unit vectors **i**, **j**, and **k**. The angles α, β, and γ are the **direction angles of v**, and cos α, cos β, and cos γ are **the direction cosines of v**.

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}$$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|}$$

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

E. Definition of Projection and Vector Components

Let **u** and **v** be nonzero vectors. Moreover, let $\mathbf{u} = \mathbf{w_1}+\mathbf{w_2}$, where $\mathbf{w_1}$ is parallel to **v** and $\mathbf{w_2}$ is orthogonal to **v.**

1. w1 is called **the projection of u onto v** or the **vector component of u along v**, and is denoted by $\mathbf{w_1} = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w_2} = \mathbf{u}-\mathbf{w_1}$ is called the **vector component of u orthogonal to v.**

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F. Theorem 10.6: Projection Using the Dot Product

If **u** and **v** are nonzero vectors, then the projection of **u** onto **v** is given by

$$proj_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

The projection of **u** onto **v** can be written as a scalar multiple of a unit vector in the direction of **v**. The scalar is called the **component of u in the direction of v**.

G. Definition of Work

The work W done by a constant force F as its point of application moves along vector PQ is given by either of the following.

1. $W = \left\|proj_{\overline{PQ}}\mathbf{F}\right\|\left\|\overline{PQ}\right\|$ Projection Form
2. $W = \mathbf{F}\bullet\overrightarrow{PQ}$ Dot Product From