10.3: The Dot Product of Two Vectors

A. Definition of Dot Product

The dot product of \( \mathbf{u} = \langle u_1, u_2 \rangle \) is \( \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 \)

The dot product of \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) is \( \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \)

Note: The dot product of two vectors yields a scalar, it is also called the inner product or scalar product of two vectors.

B. Theorem 10.4: Properties of the Dot Product

1. \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
2. \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \)
3. \( c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v} \)
4. \( \mathbf{0} \cdot \mathbf{v} = 0 \)
5. \( \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \)

C. Theorem 10.5: Angle Between Two Vectors

If \( \theta \) is the angle between two nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \), then

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}
\]

This can be rewritten as \( \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos \theta \) to find the dot product if \( \theta \) is know.

Note: The terms perpendicular and orthogonal and normal all mean essentially the same thing. However, we usually say two vectors are orthogonal, two lines are perpendicular, and a vector is normal to a given line or plan.
D. In space it is convenient to measure direction in terms of the angles between the nonzero vector \( v \) and the three unit vectors \( i, j, \) and \( k \). The angles \( \alpha, \beta, \) and \( \gamma \) are the direction angles of \( v \), and \( \cos \alpha, \cos \beta, \) and \( \cos \gamma \) are the direction cosines of \( v \).

\[
\cos \alpha = \frac{v_1}{\|v\|} \\
\cos \beta = \frac{v_2}{\|v\|} \\
\cos \gamma = \frac{v_3}{\|v\|}
\]

E. Definition of Projection and Vector Components

Let \( u \) and \( v \) be nonzero vectors. Moreover, let \( u = w_1 + w_2 \), where \( w_1 \) is parallel to \( v \) and \( w_2 \) is orthogonal to \( v \).

1. \( w_1 \) is called the projection of \( u \) onto \( v \) or the vector component of \( u \) along \( v \), and is denoted by \( w_1 = \text{proj}_v u \).
2. \( w_2 = u - w_1 \) is called the vector component of \( u \) orthogonal to \( v \).

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F. Theorem 10.6: Projection Using the Dot Product

If \( u \) and \( v \) are nonzero vectors, then the projection of \( u \) onto \( v \) is given by

\[
\text{proj}_v u = \left( \frac{u \cdot v}{\|v\|^2} \right) v
\]

The projection of \( u \) onto \( v \) can be written as a scalar multiple of a unit vector in the direction of \( v \). The scalar is called the component of \( u \) in the direction of \( v \).

G. Definition of Work

The work \( W \) done by a constant force \( F \) as its point of application moves along vector \( PQ \) is given by either of the following.

1. \( W = \| \text{proj}_{PQ} F \| \|PQ\| \) Projection Form
2. \( W = F \cdot \overrightarrow{PQ} \) Dot Product Form