Section 9.5: Area and Arc Length in Polar Coordinates

A. Theorem 9.13 – Area in Polar Coordinates

If f is continuous and nonnegative on the interval [α,β], $0 < \beta - \alpha \le 2\pi$, then the area of the region bounded by $r = f(\theta)$ between the radial lines θ = α and is θ = β

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \quad , \; 0 < \beta - \alpha \le 2\pi$$

- Helpful Identities:

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

Examples: 2, 4, 10

B. Finding points of intersection of two polar equations.

1. Set each equation solved for r equal to each other.
2. To find all points of intersection you will probably have to do more

    - Using a negative r and θ +π for θ, rewrite one of the equations and solve again.

- Analyze the graph and see if they pass through the pole. (0, 0) is a pt of intersection if they both pass through the pole.

3. Be careful when finding the corresponding r values.

Examples: 14, 18, (30)

C. Theorem 9.14 – Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval [α,β]. The length of the graph of $r = f(\theta)$ from θ = α and is θ = β is

$$s = \int_{\alpha}^{\beta} \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$$

Example: 44

D. Theorem 9.15 – Area of a Surface of Revolution

Let f be a function whose derivative is continuous on an interval [α,β]. The area of the surface formed by revolving the graph of $r = f(\theta)$ from θ = α and θ = β about the indicated line is

1. About polar axis: $S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, d\theta$
2. About the line θ = π/2:

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, d\theta$$

Example: 52