Section 8.8: Power Series

A. Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

is called a **power series**. More generally, series in the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \ldots + a_n (x-c)^n + \ldots$$

is called a **power series centered at c**, where c is a constant.

B. Theorem 8.20 – Convergence of a Power Series

The domain of a power series can take three basic forms. For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
   - The number R is the radius of convergence of the power series. If the series converges only at c, the radius of convergence is R = 0. No interval of convergence

2. There exists a real number R>0 such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$
   - The radius of convergence is R. The interval of convergence is (c-R, c+R). The endpoints must be tested separately.

3. The series converges absolutely for all x.
   - If the series converges for all x, the radius of convergence is $R = \infty$. The interval of convergence is $(-\infty,\infty)$

Examples: 6, 10

C. Theorem 8.21: Properties of Functions Defined By Power Series

If a function is given by

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$

has a radius of convergence of R>0, then, on the interval (c – R, c + R), f is differentiable . Moreover, the derivative and antiderivative of f are as follows.

1. $f\,'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots$
2. $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1} = C + a_0(x-c) + \frac{a_1(x-c)^2}{2} + \frac{a_2(x-c)^3}{3} + \ldots$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original series. The interval of convergence may differ at the endpoints.

Examples: 12, 14, 22